G_2–instantons, associative submanifolds and Fueter sections

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Abstract

We give a sufficient condition for an associative submanifold in a G_2–manifold to appear as the bubbling locus of a sequence of G_2–instantons, related to the existence of a Fueter section of a bundle of ASD instanton moduli spaces over said submanifold.

1 Introduction

Recall that a G_2*–manifold* is a 7–manifold (Y, ϕ) equipped with a torsion-free G_2–structure, that is, a non-degenerate 3–form ϕ which induces a metric (and an orientation) on Y with respect to which it is parallel. A connection A on a G–bundle over Y is called a G_2*–instanton* if it satisfies

$$*(F_A \wedge \phi) = -F_A.$$

Donaldson–Thomas [DT98] noticed that G_2–instantons are in many ways similar to flat connections on 3–manifolds. In particular, it is expected that one can construct an analogue of the Casson invariant by counting G_2–instantons.

A major difficulty in the study of gauge theory on G_2–manifolds is that moduli spaces of G_2–instantons need not be compact. Suppose that Y is a fixed compact 7–manifold together with a sequence of torsion-free G_2–structures ϕ_i converging to a torsion-free G_2–structure ϕ_∞ and suppose that A_i is a sequence of G_2–instantons on a fixed bundle E over (Y, ϕ_i). From the work of Uhlenbeck [Uhl82], Price [Pri83] and Nakajima [Nak88] it follows that there is a closed subset P of Y of finite 3–dimensional Hausdorff measure and a G_2–instanton A_∞ over $(Y \setminus P, \phi_\infty)$ such that up to gauge transformations a subsequence of A_i converges to A_∞ in C^∞_{loc} on the complement of P. This non-compactness phenomenon has two causes: the formation of non-removable singularities and bubbling in codimension four. Tian [Tia00] studied the bubbling behaviour and was able to show that P is rectifiable, that its tangent spaces are calibrated by ϕ_∞ and that to almost every point of P one can attach a bubbling tree of ASD instantons on S^4 bubbling off transversely to P. In the simplest case the singularities in A_∞ are removable, P is an *associative submanifold*, i.e., a submanifold of Y calibrated with respect to ϕ_∞ and the bubbling trees of ASD instantons consist of single ASD instantons forming a smooth section $\mathfrak{I}$ of an *instanton bundle* $\mathfrak{M}$ (associated to the normal bundle of P and the restriction to P of the bundle underlying A_∞, see Section 3).

In order to obtain a meaningful compactification of the moduli space of G_2-instantons it is important to understand when a given triple $(A_\infty, P, \mathfrak{I})$ arises from a sequence of G_2–instantons bubbling along P. Donaldson–Segal [DS11] argued that $\mathfrak{I}$ should be a *Fueter section*. A section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ is called a Fueter section if it

satisfies a certain non-linear Dirac equation associated to ϕ_∞ and the restriction of A_∞ to P called the *Fueter equation*, see Section 3. In this paper we prove that this condition is indeed sufficient under additional genericity assumptions which will be made precise in Sections 2 and 3.

Theorem 1.1. *Let Y be a compact 7–manifold equipped with a family of torsion-free G_2–structures $(\phi_t)_{t\in(-T,T)}$. Let A_0 be an unobstructed G_2–instanton on a G–bundle E_0 over (Y,ϕ_0), let P_0 be an unobstructed associative submanifold in (Y,ϕ_0), let $\mathfrak{M}_0$ be an instanton bundle associated to P_0 and $E_0|_{P_0}$ and let $\mathfrak{I}_0$ be a Fueter section of $\mathfrak{M}_0$ which is unobstructed with respect to (ϕ_t). Then there is a constant $\Lambda > 0$, a G–bundle E together with a family of connections $(A_\lambda)_{\lambda\in(0,\Lambda]}$ and a continuous function $t\colon [0,\Lambda] \to (-T,T)$ such that $t(0) = 0$ and A_λ is a G_2–instanton on $(Y,\phi_{t(\lambda)})$ for all $\lambda \in (0,\Lambda]$. Moreover, as λ goes to zero A_λ converges to A_0 on the complement of P_0 and at each point $x \in P_0$ an ASD instanton in the equivalence class given by $\mathfrak{I}(x)$ bubbles off transversely.*

This results should be seen as a first step towards making the picture outlined by Donaldson–Segal [DS11] rigorous. Unfortunately, there are at the moment no known examples of the input required by our result to produce a bubbling sequence of G_2–instantons. As more constructions of associative submanifolds and G_2–instantons become available in the future we expect this situation to improve.

The analysis involved in the proof of Theorem 1.1 is an extension of that required for the construction of G_2–instantons on generalised Kummer constructions in [Wal11]. As such there are some similarities with the Lewis' construction of Spin(7)–instantons [Lew98], unpublished work by Brendle [Bre03] on Spin(7)–instantons and Pacard–Ritoré's work on the Allen–Cahn equation [PR03].

We should point our here that there are suitably adapted versions of the above result pertinent to the study of gauge theory on Calabi–Yau 3–folds and Spin(7)–manifolds. In particular, this should play a role in an analytic construction of Donaldson–Thomas invariants for Calabi–Yau 3–folds.

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2 Review of geometry on G_2–manifolds

Let V be a 7–dimensional vector space equipped with a non-degenerate 3–form ϕ. Here by non-degenerate we mean that for each non-zero vector $v \in V$ the 2–form $i(v)\phi$ on the quotient is $V/\langle v\rangle$ is symplectic. Then V carries a unique inner product g and orientation such that

$$i(v_1)\phi \wedge i(v_2)\phi \wedge \phi = 6g(v_1,v_2)\mathrm{vol}$$

for all $v_1, v_2 \in V$. It is well-known, see [SW10, Theorem 3.2], that any two non-degenerate forms on V are related by an automorphism of V. In particular, an appropriate choice of basis identifies ϕ with the model

$$\phi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}.$$

Here $\mathrm{d}x^{ijk}$ is a shorthand for $\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ and $x_1,\dots,x_7$ are standard coordinates on $\mathbf{R}^7$. The stabiliser of ϕ_0 in GL(7) is known to be isomorphic to the exceptional Lie group G_2.

Definition. A G_2*–manifold* manifold is a 7–manifold Y equipped with a torsion-free G_2–structure, i.e., a non-degenerate 3–form ϕ which is parallel with respect to the metric it induces on Y. We call ϕ the *associative calibration* and

$$\psi := *\phi$$

the *coassociative calibration*.

G_2–manifolds have holonomy contained in G_2, hence the name. Moreover, they are spin and admit (at least) one non-trivial parallel spinor and are therefore Ricci-flat.

Examples. Compact examples of G_2–manifolds can be build out of hyperkähler surfaces by taking the product with T^3 or out of Calabi–Yau 3–fold by taking the product with S^1. In both cases the holonomy group is strictly contained in G_2. Compact examples with full holonomy G_2 can be produced via Joyce's generalised Kummer construction [Joy96] and Kovalev's twisted connected sum construction [Kov03].

2.1 G_2–instantons on G_2–manifolds

Let (Y, ϕ) be a compact G_2–manifold and let E be a G–bundle over Y where G is a compact Lie group, say $G = \mathrm{SO}(3)$ or $G = \mathrm{SU}(2)$.

Definition. A connection A on E is called a G_2*–instanton* on (Y, ϕ) if it satisfies

$$*(F_A \wedge \phi) = -F_A. \tag{2.1}$$

Since ϕ is closed it follows from the Bianchi identity that G_2–instantons are Yang–Mills connections. In fact, there is an energy identity which shows that G_2–instantons are absolute minima of the Yang–Mills functional.

Examples. On G_2–manifolds arising from hyperkähler surfaces (resp. Calabi–Yau 3–folds) pullbacks of ASD instantons (resp. Hermitian–Yang–Mills connections) are G_2–instantons. The Levi–Civita connection of a G_2–manifold is a G_2–instanton. The first non-trivial examples G_2–instantons with $G = \mathrm{SO}(3)$ where recently constructed by the author in [Wal11].

It is interesting to note that equation (2.1) is equivalent to

$$F_A \wedge \psi = 0. \tag{2.2}$$

To see this one uses the identity $\omega + *(\omega \wedge \phi) = *(\psi \wedge *(\psi \wedge \omega))$ for 2–forms ω on Y. From Equation (2.2) one can see that G_2–instantons are in many ways similar to flat connections on 3–manifolds. In particular, if A_0 is a G_2–instanton, then there is a G_2–Chern–Simons functional

$$CS^\psi(A_0 + a) = \int_Y \left\langle \mathrm{d}_{A_0} a \wedge a + \frac{1}{3}[a \wedge a] \wedge a \right\rangle \wedge \psi$$

whose critical points are precisely G_2–instantons. It is not entirely unreasonable to expect that some of the 3–manifolds invariants arising from the Chern–Simons functional, like the Casson invariant and instanton Floer homology, have G_2–analogues. This idea goes back at least to the seminal paper of Donaldson–Thomas [DT98] and is one of the main motivations for studying G_2–instantons.

From an analytical perspective equation (2.2) is slightly unsatisfactory, since $\mathrm{d}_A(F_A \wedge \psi) = 0$. It is more convenient to work with the following equivalent equation

$$F_A \wedge \psi + *\mathrm{d}_A \xi = 0 \tag{2.3}$$

where $\xi \in \Omega^0(Y, \mathfrak{g}_E)$.

Definition. The gauge fixed *linearisation* of equation (2.3) is given by the linear operator $L_A\colon \Omega^0(Y,\mathfrak{g}_E)\oplus\Omega^1(Y,\mathfrak{g}_E)\to\Omega^0(Y,\mathfrak{g}_E)\oplus\Omega^1(Y,\mathfrak{g}_E)$ defined by

$$L_A = \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi\wedge\mathrm{d}_A) \end{pmatrix}.$$

This is a self-adjoint elliptic operator and governs the infinitesimal deformation theory of the G_2–instanton A. In particular, the virtual dimension of the moduli space of G_2–instantons is zero in accordance with the fact that G_2–instantons are critical points of the G_2–Chern–Simons functional.

Definition. A G_2–instanton A is called *unobstructed* if L_A surjective.

Proposition 2.4. *Let Y be a compact 7–manifold and let $(\phi_t)_{t\in(-T,T)}$ be a family of torsion-free G_2–structures on Y. Suppose that A is an unobstructed G_2–instanton a G–bundle E over (Y,ϕ_0). Then there is a constant $T'\in(0,T]$ and a unique family of G_2–instantons $(A_t)_{t\in(-T',T')}$ on E over (Y,ϕ_t) with $A_0=A$.*

2.2 Associative submanifolds in G_2–manifolds

Let (Y,ϕ) be a compact G_2–manifold. Then ϕ is a calibration in the sense of Harvey–Lawson [HL82], i.e., ϕ is closed and for each oriented 3–dimensional subspace P of T_xY the following holds

$$\mathrm{vol}_P \le \phi|_P. \tag{2.5}$$

Definition. An oriented submanifold P is called an *associative submanifold* in (Y,ϕ) if for each $x\in P$ we have $\mathrm{vol}_{T_xP}=\phi|_{T_xP}$.

By definition, associative submanifold are calibrated with respect to ϕ and hence volume-minimising in their homology class.

Examples. In G_2–manifolds arising from Calabi–Yau 3–folds products of holomorphic curves with S^1 and special Lagrangians are examples of associative submanifolds. Associative submanifolds also arise as 3–dimensional fixed point sets of orientation reversing involutions of Y mapping ϕ to $-\phi$. For concrete examples we refer the reader to Joyce [Joy96, Part II, Section 4.2].

Remark. The importance of associative submanifolds in the study of gauge theory on G_2–manifolds is due to the following fact. Consider $(\mathbf{R}^7,\phi_0)$ and an orthogonal decomposition $\mathbf{R}^7=\mathbf{R}^3\oplus\mathbf{R}^4$. Let I be a connection on a bundle over $\mathbf{R}^4$. Then the pullback of I to $\mathbf{R}^7$ is a G_2–instanton if and only if there is an orientation on $\mathbf{R}^3$ with respect to which it is calibrated by ϕ_0 and I is an ASD instanton on $\mathbf{R}^4$. This is the reason why the bubbling locus of a sequence of G_2–instantons is associative and why ASD instantons bubble off transversely.

In the following we will require some results due to McLean [McL98] about the deformation theory of associative submanifolds. If P is an associative submanifold then there is a natural identification

$$TP \cong \Lambda^+N^*P \tag{2.6}$$

given by inserting tangent vectors to P into ϕ (and restricting to NP). Thinking of Λ^+N^*P as a sub-bundle of $\mathfrak{so}(NP)$ this yields a Clifford multiplication $\gamma: TP\to\mathrm{End}(NP)$. Denote by $\bar\nabla$ the connection on NP induced by the Levi–Civita connection on Y.

Definition. The *Fueter operator* $F_P\colon \Gamma(NP) \to \Gamma(NP)$ associated to P is defined by

$$F_P(n) := \sum_i \gamma(e_i)\bar{\nabla}_i n.$$

Remark. The Fueter operator F_P can be identified with a twisted Dirac operator as follows. Pick a spin structure $\mathfrak{s}$ on P. Because of the identification (2.6) there is a unique $\mathrm{SU}(2)$–bundle $\mathfrak{u}$ over P such that $\mathfrak{s}\times\mathfrak{u}$ is a spin structure on NP. The bundle $\mathfrak{u}$ also comes with a connection, such that the resulting connection on $\mathfrak{s}\times\mathfrak{u}$ is a spin connection. If S and U denote the quaternionic line bundles corresponding to $\mathfrak{s}$ and $\mathfrak{u}$, then $S\otimes_{\mathbf{C}} U$ has a natural real structure and its real part can be identified with NP. With respect to this identification F_P becomes the twisted Dirac operator $D\colon \Gamma(\Re(S\otimes_{\mathbf{C}} U)) \to \Gamma(\Re(S\otimes_{\mathbf{C}} U))$.

The importance of F_P is that it controls the infinitesimal deformation theory of the associative submanifold P. In particular, the moduli space of associative submanifold near P is modelled on the zero set of a smooth map from the kernel of F_P to its cokernel.

Definition. An associative submanifold P is called *unobstructed* if F_P is surjective.

Proposition 2.7 (McLean [McL98])**.** *Let Y be a compact 7–manifold and let $(\phi_t)_{t\in(-T,T)}$ be a family of torsion-free G_2–structures on Y. Suppose that P is an unobstructed associative submanifold in (Y,ϕ_0). Then there is a constant $T'\in(0,T]$ and a unique family of associative submanifolds $(P_t)_{t\in(-T',T')}$ in (Y,ϕ_t) with $P_0=P$.*

3 Fueter sections of instanton moduli bundles

Let us begin by recalling some facts about moduli spaces of framed finite energy ASD instantons on $\mathbf{R}^4$. Fix a G–bundle E over $S^4=\mathbf{R}^4\cup\{\infty\}$. Denote by M the moduli space of ASD instantons on E framed over the point at infinity. By Uhlenbeck's removable singularities theorem we can think of M as a moduli space of finite energy ASD instantons on $\mathbf{R}^4$ framed at infinity. The infinitesimal deformation theory of such an ASD instanton I is governed by the linear operator $\delta_I\colon \Omega^1(\mathbf{R}^4,\mathfrak{g}_E)\to\Omega^0(\mathbf{R}^4,\mathfrak{g}_E)\oplus\Omega^+(\mathbf{R}^4,\mathfrak{g}_E)$ defined by

$$\delta_I a := (\mathrm{d}_I^* a, \mathrm{d}_I^+ a). \tag{3.1}$$

In a suitable functional analytic setup it can be seen that δ_I is always surjective and, moreover, that the kernel of δ_I lies in L^2, see Taubes [Tau83] or Proposition 5.3. This implies that M is a smooth manifold and can be equipped with the L^2 metric arising from the standard metric on $\mathbf{R}^4$. Note that $\Lambda^+ := \Lambda^+(\mathbf{R}^4)^*\cong\mathfrak{so}(4)$ acts on $\mathbf{R}^4$ as well as on $\mathbf{R}\oplus\Lambda^+$. It is easy to see that the operator δ_I commutes with the induced action of Λ^+ and hence we obtain an $\mathrm{SO}(4)$–equivariant action of Λ^+ on TM. Moreover, M has carries an action of $\mathbf{R}^4\rtimes\mathbf{R}^+$ where $\mathbf{R}^4$ acts by translation and $\mathbf{R}^+$ acts by dilation, i.e., by pullback via s_λ where $s_\lambda(x)=x/\lambda$ for $\lambda\in\mathbf{R}^+$. Since the centre of mass of the measure $|F_I|^2\,\mathrm{dvol}$ is equivariant with respect to the $\mathbf{R}^4$–action, we can write $M=\mathring{M}\times\mathbf{R}^4$ where $\mathring{M}$ is the space of instantons centred at zero.

Now, let (Y,ϕ) be a G_2–manifold and let P be an associative submanifold in Y. Fix a moduli space M of framed finite energy ASD instantons on $\mathbf{R}^4$ as above and let E_∞ be a G–bundle over P together with a connection A_∞.

Definition. We define the *instanton bundle* $\mathfrak{M}$ associated to P, M and E_∞ by

$$\mathfrak{M} := (\mathrm{Fr}(NP)\times E_\infty)\times_{\mathrm{SO}(4)\times G} M.$$

We will apply this construction with E_∞ being the restriction to P of the bundle supporting the background G_2–instanton into which we want to graft a section of $\mathfrak{M}$.

Example. The moduli space of framed charge one SU(2)–instantons on $\mathbf{R}^4$ is isomorphic to $(S^+ \setminus \{0\})/\mathbf{Z}_2 \times \mathbf{R}^4$, where S^+ is the positive spin representation of $\mathbf{R}^4$. If we pick $\mathfrak{s}$ and $\mathfrak{u}$ as in the remark in Section 2.2, then

$$\mathfrak{M} = (\mathfrak{s} \times \mathfrak{u} \times E_\infty) \times_{\mathrm{Spin}(4)\times \mathrm{SU}(2)} M = (\Re(S \otimes E_\infty) \setminus \{0\})/\mathbf{Z}_2 \times NP.$$

Denote by $N_\infty P := \mathrm{Fr}(NP) \times_{\mathrm{SO}(4)} S^4$ the sphere-bundle obtained from NP by adjoining a section at infinity.

Theorem 3.2 (Donaldson–Segal [DS11] and Haydys [Hay11])**.** *To each section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ we can associate a G–bundle $E = E(\mathfrak{I})$ over $N_\infty P$ together with a connection $I = I(\mathfrak{I})$ and a framing $\Phi\colon E|_\infty \to E_\infty$ such that:*

- *For each $x \in P$ the restriction of I to $N_x P$ represents $\mathfrak{I}(x)$.*
- *The framing Φ identifies the restriction of I to the section at infinity with A_∞.*

The actions of $\mathbf{R}^+$ on $\mathbf{R}^4$ and M lifts to a fibre-wise actions on NP and $\mathfrak{M}$. We can assume with out loss of generality that the construction in Theorem 3.2 is equivariant with respect to the action of $\mathbf{R}^+$, that is, $I(s_\lambda^*\mathfrak{I}) = s_\lambda^* I(\mathfrak{I})$. It will be convenient to use the shorthand notations $I_\lambda := I(s_\lambda^*\mathfrak{I})$ and $\mathfrak{I}_\lambda := s_\lambda^*\mathfrak{I}$.

If a section $\mathfrak{I}$ does arise from a sequence of G_2–instantons bubbling along P, then it is reasonable to expect that in the limit as λ goes to zero the connection I_λ is close to being a G_2–instanton. To make sense of that notion we define the 4–form ψ_0 on NP to be the zeroth order Taylor expansion of ψ off P. More explicitly, we can write ψ_0 as

$$\psi_0 := e^1 \wedge e^2 \wedge \omega_{e_3} + e^2 \wedge e^3 \wedge \omega_{e_1} + e^3 \wedge e^1 \wedge \omega_{e_2} + \mathrm{vol}_{NP}. \tag{3.3}$$

Here (e_i) is a local positive orthonormal frame on P, (e^i) is its dual frame, vol_{NP} is the fibre-wise volume form on NP and $v \in TP \mapsto \omega_v \in \Lambda^+ N^*P$ is given by the identification (2.6). With this notation set up the natural requirement is that

$$\lim_{\lambda\to 0} \lambda^2 F_{I_\lambda} \wedge \psi_0 = 0. \tag{3.4}$$

If we introduce a bi-grading on k–forms on NP according to the splitting $TNP = \pi^*TP \oplus \pi^*NP$ corresponding to the connection on NP with $\pi\colon NP \to P$ being the canonical projection, then it is easy to see that equation (3.4) splits into two parts. The first one is simply the condition that the anti-self-dual part of $F_I^{0,2}$ must vanish, while the second part is given by

$$F_I^{1,1} \wedge (\psi_0 - \mathrm{vol}_{NP}) = 0.$$

This condition can be understood as a partial differential equation on $\mathfrak{I}$ as follows. Define the vertical tangent bundle $V\mathfrak{M}$ to $\mathfrak{M}$ by

$$V\mathfrak{M} := (\mathrm{Fr}(NP) \times E_\infty) \times_{\mathrm{SO}(4)\times G} TM.$$

If $\mathfrak{I}$ is a section of $\mathfrak{M}$, then the action of Λ^+ on M induces a Clifford multiplication $\gamma : TP \to \mathrm{End}(\mathfrak{I}^*V\mathfrak{M})$ in view of the identification (2.6). Moreover, the connections on NP and E_∞ induce a connection ∇ on $\mathfrak{M}$ associating to each section $\mathfrak{I}$ its covariant derivative $\nabla\mathfrak{I} \in \Omega^1(\mathfrak{I}^*V\mathfrak{M})$.

Definition. The *Fueter operator* $\mathfrak{F}$ associated to $\mathfrak{M}$ is defined by

$$\mathfrak{I} \in \Gamma(\mathfrak{M}) \mapsto \mathfrak{F}\mathfrak{I} := \sum_i \gamma(e_i)\nabla_i \mathfrak{I} \in \Gamma(\mathfrak{I}^*\mathfrak{M})$$

where (e_i) is a local orthonormal frame on P. A section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ is called a *Fueter section* if it satisfies $\mathfrak{F}\mathfrak{I} = 0$.

Example (continued)**.** If M is the moduli space of charge one SU(2)–instantons, then the Fueter operator $\mathfrak{F}$ is lifts to the twisted Dirac operator $D\colon \Gamma(\Re(S \otimes_{\mathbf{C}} (E_\infty \oplus U)) \to \Gamma(\Re(S \otimes_{\mathbf{C}} (E_\infty \oplus U))$.

Remark. The Fueter operator $\mathfrak{F}$ is compatible with the product structure on $\mathfrak{M} = \mathring{\mathfrak{M}} \times NP$ corresponding to $M = \mathring{M} \times \mathbf{R}^4$. Its restriction to the second factor is given by the Fueter operator F_P associated to P.

Theorem 3.5 (Donaldson–Segal [DS11] and Haydys [Hay11])**.** *If $\mathfrak{I} \in \Gamma(\mathfrak{M})$, then we can identify $\Gamma(\mathfrak{I}^*V\mathfrak{M})$ with a subspace of $\Omega^1\left(NP, \mathfrak{g}_{E(\mathfrak{I})}\right)$. With respect to this identification we have the following identity*

$$\mathfrak{F}\mathfrak{I} = *_0\left(F_{I(\mathfrak{I})} \wedge (\psi_0 - \mathrm{vol}_{NP})\right)$$

where $*_0$ is the Hodge–$*$–operator on NP. In particular, $I(\mathfrak{I})$ satisfies equation (3.4) if and only if $\mathfrak{I}$ is a Fueter section.

Definition. The *linearised Fueter operator* $F_{\mathfrak{I}}\colon \Gamma(\mathfrak{I}^*V\mathfrak{M}) \to \Gamma(\mathfrak{I}^*V\mathfrak{M})$ for $\mathfrak{I} \in \Gamma(\mathfrak{M})$ defined by

$$F_{\mathfrak{I}}(\hat{\mathfrak{I}}) = \sum_i \gamma(e_i)\nabla_i \hat{\mathfrak{I}}$$

where (e_i) is a local orthonormal frame on P.

If $\mathfrak{I}$ is a Fueter section, then even though $F_{\mathfrak{I}}$ is self-adjoint and thus has index zero it will never be invertible. This is because Fueter sections come in 1–parameter families $s_\lambda^*\mathfrak{I}$. In particular, taking the derivative at $\lambda = 1$ yields an element in the kernel of $F_{\mathfrak{I}}$. If $\hat{v} \in \Gamma(V\mathfrak{M})$ denotes the vector field generating the action of $\mathbf{R}^+$ on $\mathfrak{M}$, then we can succinctly write this element of the kernel as $\hat{v} \circ \mathfrak{I}$.

Let (ϕ_t) be a family of torsion-free G_2–structures on Y, let A_0 be an unobstructed G_2–instanton on a G–bundle E over (Y, ϕ_0) and let P_0 be an unobstructed associative submanifold on (Y, ϕ_0). Then by Propositions 2.4 and 2.7, we obtain a family of a family of G_2–instantons (A_t) over (Y, ϕ_t) and a family of associative submanifolds (P_t) in (Y, ϕ_t). Now carry out the above construction with $P = P_t$, $E_\infty = E|_{P_t}$, $A_\infty = A_t|_{P_t}$ and a fixed moduli space M of ASD instantons on some G–bundle over S^4 to obtain a family of instanton bundles $(\mathfrak{M}_t)$ along with a family of Fueter operators $(\mathfrak{F}_t)$. If $\mathfrak{I}_0$ is a Fueter section of $\mathfrak{M}_0$ with

$$\dim\ker F_{\mathfrak{I}} = 1,$$

then we obtain a family $(\mathfrak{I}_t)$ of sections of $\mathfrak{M}_t$ satisfying

$$\mathfrak{F}_t\mathfrak{I}_t + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_t = 0 \tag{3.6}$$

where μ is a smooth function vanishing at zero.

Definition. In the above situation we say that $\mathfrak{I}_0$ is *unobstructed* with respect to (ϕ_t) if

$$\left.\frac{\partial\mu}{\partial t}\right|_{t=0} \neq 0.$$

Example (continued)**.** If M is the moduli space of charge one SU(2)–instantons, then equation (3.6) can be viewed as the spectral flow of a family of twisted Dirac operator and $\mathfrak{I}_0$ is unobstructed if and only if this spectral flow has a regular crossing at $\mathfrak{I}_0$.

4 Overview of the proof of Theorem 1.1

We will now begin the proof of Theorem 1.1 with a rough overview and a discussion of the main difficulties. Along the line we will carry out a few minor steps of the argument and set up some notation that will be used later on. Suppose $(\phi_t)_{t\in(-T,T)}$, A_0, P_0 and $\mathfrak{I}_0 \in \Gamma(\mathfrak{M}_0)$ are as in the hypothesis of Theorem 1.1.

Convention. We fix constants $T' \in (0,T]$ and $\Lambda > 0$ such that all of the statements of the kind "if $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$, then ..." appearing in the following are valid. This is possible since there is only a finite number of these statements and each one of them is valid provided T' and Λ are sufficiently small. By $c > 0$ we will denote a "generic constant" whose value depends neither on $t \in (-T',T')$ nor on $\lambda \in (0,\Lambda]$ buy may change from one line to the other.

As discussed at the end of Section 3, A_0 and P_0 give rise to families $(A_t)_{t\in(-T',T')}$ of G_2–instantons over (Y,ϕ_t) and associative submanifolds $(P_t)_{t\in(-T',T')}$ in (Y,ϕ_t). From these we obtain a family of instanton bundles $(\mathfrak{M}_t)_{t\in(-T',T')}$ together with sections $(\mathfrak{I}_t)_{t\in(-T',T')}$ satisfying

$$\mathfrak{F}_t\mathfrak{I}_t + \mu(t)\hat{v}\circ\mathfrak{I}_t = 0 \tag{4.1}$$

where $\mu : (-T',T') \to \mathbf{R}$ is a smooth function vanishing at zero with

$$\left.\frac{\partial\mu}{\partial t}\right|_{t=0} \neq 0.$$

Our proof of Theorem 1.1 proceeds via a gluing construction. As a first step we explain how to construct approximate solutions.

Proposition 4.2. *For each $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$ we can explicitly construct a G–bundle $E_{t,\lambda}$ together with a connection $A_{t,\lambda}$ from (E,A_t) and $\mathfrak{I}_t$. The bundles $E_{t,\lambda}$ are pairwise isomorphic.*

Before we embark on the proof, let us set up some notation. Fix a constant $\sigma > 0$ such that for all $t \in (-T',T')$ the exponential map identifies a tubular neighbourhood of width σ of P_t in Y with a neighbourhood of the zero section in NP_t. For $I \subset \mathbf{R}$ we define

$$U_{I,t} := \{v \in NP_t : |v| \in I\} \qquad V_{I,t} := \{x \in Y : r_t(x) \in I\}.$$

Fix a smooth-cut off function $\chi\colon [0,\infty) \to [0,1]$ which vanishes on $[0,1]$ and is equal to one on $[2,\infty)$. For $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$ we define $\chi^-_{t,\lambda}\colon Y \to [0,1]$ and $\chi^+_t\colon Y \to [0,1]$ by

$$\chi^-_{t,\lambda}(x) := \chi(r_t(x)/2\lambda) \qquad \chi^+_t(x) := 1 - \chi(r_t(x)/2\sigma)$$

where $r_t(x) := d(x,P_t)$.

Proof of Proposition 4.2. Using radial parallel transport we can identify $E(\mathfrak{I}_t)$ over $U_{(R,\infty),t}$ for some $R > 0$ with the pullback of $E(\mathfrak{I}_t)|_\infty$ to said region, and similarly we can identify E_t over $V_{[0,\sigma),t}$ with the pullback of $E_t|_{P_t}$. Hence, via the framing

Φ we can identify $s_\lambda^* E(\mathfrak{I}_t)$ with E_t over $V_{(\lambda,\sigma),t}$. Patching both bundles via this identification yields $E_{t,\lambda}$.

To construct a connection on $E_{t,\lambda}$ note that on the overlap $I_{t,\lambda} := s_\lambda^* I(\mathfrak{I}_t)$ and A_t can be written as

$$
\begin{aligned}
A_t &= A_t|_{P_t} + a_t \\
I_{t,\lambda} &= A_t|_{P_t} + i_{t,\lambda}
\end{aligned}
\tag{4.3}
$$

where by a slight abuse of notation we denote by $A_t|_{P_t}$ the pullback of $A_t|_{P_t}$ to the overlap. We define $A_{t,\lambda}$ by interpolating between I_λ and A_t on the overlap as follows

$$
A_{t,\lambda} := A_t|_{P_t} + \chi_{t,\lambda}^- a_t + \chi_t^+ i_{t,\lambda}. \tag{4.4}
$$

□

Remark. The precise way in which we interpolate in equation (4.4) will turn out to be rather important, since it minimises the error introduced by derivatives of the cut-off functions, when measured at the appropriate scale, and our construction crucially hinges on this error to be quite small.

Now the task at hand is to solve the equation

$$
*\left(F_{A_{t,\lambda}+a} \wedge \psi_t\right) + \mathrm{d}_{A_{t,\lambda}+a}\xi = 0 \tag{4.5}
$$

where $t = t(\lambda)$, $a = a(\lambda)$ and $\xi = \xi(\lambda)$. If we could find an appropriate analytic setup in which $(a,\xi) = 0$ becomes closer and closer to being a solution of equation (4.5) while at the same time the linearisations $L_{t,\lambda} := L_{A_{t,\lambda}}$ possess right inverses which can be controlled uniformly in t and λ, then it would not be too difficult to solve equation (4.5) for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$. But, since the properties of $L_{t,\lambda}$ is closely linked among other things to those of $F_{\mathfrak{I}_t}$ and $F_{\mathfrak{I}_0}$ has a one-dimensional cokernel, we will only be able to solve equation (4.5) "modulo the cokernel of $F_{\mathfrak{I}_0}$". It turns out that the one-dimensional residual equation can be identified with

$$
\mu(t) + \eta(t,\lambda) = 0
$$

where η and $\partial_t \eta$ go to zero as λ goes to zero. Since $\partial_t \mu(0) \neq 0$ finding $t = t(\lambda)$ such that equation (4.5) is satisfied is then a simple consequence of the implicit function theorem.

Let us now discuss some aspects of the analysis. First of all we will introduce appropriate weighted Hölder spaces in Section 5. One should think of these weighted spaces as a convenient framework to deal with different local scales simultaneously. In our case they are constructed to counteract the fact that the curvature of the connection $A_{t,\lambda}$ around P_t becomes larger and larger as λ goes to zero. We will see in Section 6 that the amount by which our approximate solutions $A_{t,\lambda}$ fail to be solutions of equation (4.5) "modulo the cokernel of $F_{\mathfrak{I}_0}$" measured in our weighted Hölder norms goes to zero at a certain rate as λ goes to zero. The key difficulty lies in analysing the linearisation $L_{t,\lambda}$. As is the case in most adiabatic limit constructions $L_{t,\lambda}$ is rather badly behaved rather badly behaved on an infinite dimensional space. To make this more precise we define a map $\iota_{t,\lambda}\colon \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t) \to \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}})$ by

$$
\iota_{t,\lambda}\hat{\mathfrak{I}} := \chi_t^+ s_\lambda^* \hat{\mathfrak{I}}.
$$

Here we first identify $\hat{\mathfrak{I}} \in \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ with an element of $\Omega^1(NP, E(\mathfrak{I}_t))$, then consider the restriction of its pullback via s_λ to $U_{[0,\sigma),t}$ as an element of $\Omega^1(V_{[0,\sigma),t}, \mathfrak{g}_{E_{t,\lambda}})$

and finally extended it to all of Y by multiplication with χ_t^+. Now one can see that the appropriate norm of $i_{t,\lambda}\hat{\mathfrak{I}}$ is essentially independent of λ, while the appropriate norm of $L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}}$ goes to zero as λ goes to zero. To overcome this issue it is convenient to split the problem at hand into a part coming from the $\Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ and the part "orthogonal" to it. We define $\pi_{t,\lambda} : \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}}) \to \Gamma(V\mathfrak{M}_t)$ by

$$(\pi_{t,\lambda} a)(x) = \sum_\kappa \int_{N_x P} \langle a, \iota_{t,\lambda}\kappa\rangle\, \kappa$$

for $x \in P_t$. Here κ runs through an orthonormal basis of $(V\mathfrak{M}_t)_{\mathfrak{I}(x)}$ with respect to the inner product $\langle \iota_{t,\lambda}\cdot, \iota_{t,\lambda}\cdot\rangle$. Clearly $\pi_{t,\lambda}\iota_{t,\lambda} = \mathrm{id}$ and hence $\iota_{t,\lambda}\pi_{t,\lambda}$ is a projection. We denote the complementary projection by $\rho_{t,\lambda} := \mathrm{id} - \iota_{t,\lambda}\pi_{t,\lambda}$. If we define

$$\mathfrak{A}_{t,\lambda} := \Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \ker \pi_{t,\lambda},$$

then we can write

$$\Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}}) = \mathfrak{A}_{t,\lambda} \oplus \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$$

and decompose $L_{t,\lambda}$ accordingly into

$$L_{t,\lambda} =: \begin{pmatrix} \mathfrak{L}_{t,\lambda} & \\ & \mathfrak{K}_{t,\lambda} \end{pmatrix} + \begin{pmatrix} & \mathfrak{p}_{t,\lambda} \\ \mathfrak{q}_{t,\lambda} & \end{pmatrix}.$$

We will see in Section 7, that $\mathfrak{L}_{t,\lambda}$ has uniformly bounded right inverses with respect to our weighted Hölder norms. The operator $\mathfrak{K}_{t,\lambda}$ can be identified essentially with $F_{\mathfrak{I}_t}$ and we will thus be able to construct appropriate right inverses "modulo the cokernel of $F_{\mathfrak{I}_0}$" albeit not quite uniformly bounded with respect to our Hölder norms. The operators $\mathfrak{p}_{t,\lambda}$ and $\mathfrak{q}_{t,\lambda}$ will turn out to be negligibly small perturbations. In Section 8 we discuss how to control the non-linearity $Q_{t,\lambda}$ in equation (4.5) given by

$$Q_{t,\lambda}(\underline{a}, \hat{\mathfrak{I}}) := *\left(\Big[(a + \iota_{t,\lambda}\hat{\mathfrak{I}}) \wedge (a + \iota_{t,\lambda}\hat{\mathfrak{I}})\Big] \wedge \psi_t\right) + [\xi, a + \iota_{t,\lambda}\hat{\mathfrak{I}}]$$

where $\underline{a}$ is a shorthand for

$$\underline{a} = (\xi, a) \in \mathfrak{A}_{t,\lambda}.$$

Once this is done, the completion of the proof of Theorem 1.1 in Section 9 is rather straight-forward.

5 Weighted Hölder norms

For $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we define a family of weight functions $w_{\ell,\delta,t,\lambda}$ on Y depending on two additional parameters $\ell \in \mathbf{R}$ and $\delta \in \mathbf{R}$ as follows

$$w_{\ell,\delta,t,\lambda}(x) := \begin{cases} \lambda^\delta(\lambda + r_t(x))^{-\ell-\delta} & \text{if } r_t(x) \le \sqrt{\lambda} \\ r_t(x)^{-\ell+\delta} & \text{if } r_t(x) > \sqrt{\lambda} \end{cases}$$

and let $w_{\ell,\delta,t,\lambda}(x, y) := \min\{w_{\ell,\delta,t,\lambda}(x), w_{\ell,\delta,t,\lambda}(x)\}$. For a tensor field f on Y (with values in $\mathfrak{g}_{E_{t,\lambda}}$) we define weighted Hölder norms

$$\begin{aligned}
\|f\|_{L^\infty_{\ell,\delta,t,\lambda}} &:= \|w_{\ell,\delta,t,\lambda} f\|_{L^\infty} \\
[f]_{C^{0,\alpha}_{\ell,\delta,t,\lambda}} &:= \sup_{\substack{x\neq y \\ d(x,y)\le\lambda+\min\{r_t(x), r_t(y)\}}} w_{\ell-\alpha,\delta,t,\lambda}(x, y)\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \\
\|f\|_{C^{k,\alpha}_{\ell,\delta,t,\lambda}} &:= \sum_{j=0}^k \|\nabla^k f\|_{L^\infty_{\ell-j,\delta,t,\lambda}} + [\nabla^k f]_{C^{0,\alpha}_{\ell-j,\delta,t,\lambda}}
\end{aligned}$$

Here the metrics are the ones induced by the Riemannian metric on Y and the metric on $\mathfrak{g}_{E_{t,\lambda}}$ while the connection is the one induced by the Levi–Civita connection on TY and the connection $A_{t,\lambda}$. We use parallel transport with respect to this connection to compare $f(x)$ and $f(y)$ in the definition of the weighted Hölder semi-norm. It will often be convenient to use these norms on a subset, say U, of Y. We will denote the corresponding norms by $\|f\|_{C^{k,\alpha}_{\ell,\delta,t,\lambda}(U)}$.

Remark. The reader may find the following heuristic useful for understanding the above norms. Suppose f is a k–form on Y. Fix a small ball centred in P_t, identify it with a small ball in $\mathbf{R}^3 \times \mathbf{R}^4$ and rescale this ball by a factor $1/\lambda$. Upon pulling everything back to this rescaled ball the weight function $w_{-k,\delta,t,\lambda}$ becomes essentially $(1+|y|)^{k-\delta}$, where y denotes the $\mathbf{R}^4$–coordinate. Thus as λ goes to zero a uniform bound on $\|f_\lambda\|_{L^\infty_{-k,\delta,t,\lambda}}$ ensures that the pullbacks of f_λ decay like $|y|^{-k+\delta}$ in the direction of $\mathbf{R}^4$. At the same time it forces f_λ not to blowup at a rate faster than $r_t^{-k-\delta}$ along P_t. The apparent discrepancy in the exponents can be understood by considering the action of the inversion $y \mapsto \sqrt{\lambda}y/|y^2$.

A useful property of the weighted Hölder norms we just introduced is that they are compatible with multiplication in the sense that the following holds

$$\|f \cdot g\|_{C^{k,\alpha}_{\ell_1+\ell_2,\delta_1+\delta_2,t,\lambda}} \leq \|f\|_{C^{k,\alpha}_{\ell_1,\delta_1,t,\lambda}} \|g\|_{C^{k,\alpha}_{\ell_2,\delta_2,t,\lambda}}. \tag{5.1}$$

This follows directly from the definition and we will make frequent use of this fact. As a first consequence note that whenever $\delta \geq \delta'$, then

$$\|f\|_{C^{k,\alpha}_{\ell,\delta,t,\lambda}} \leq c\|f\|_{C^{k,\alpha}_{\ell,\delta',t,\lambda}}, \tag{5.2}$$

because $\|1\|_{C^{k,\alpha}_{0,\delta-\delta',t,\lambda}} \leq c$. Here as per our convention $c > 0$ is a constant independent of $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$.

We measure sections of $\mathfrak{I}_t^* V\mathfrak{M}_t$ using standard Hölder norms on P_t. In fact, we will really be working with $\lambda\|\cdot\|_{C^{k,\alpha}}$, although the additional factor of λ will not show up explicitly until Section 9. In any case, the reader should keep this factor in mind, since it will make some of our estimates appear more natural and symmetric. To better understand the relation between standard Hölder norms and the weighted Hölder norms introduced above we first recall the following well-known result whose proof can be found, e.g., in [Wal11, Proposition 5.2].

Proposition 5.3. *Let E be a G–bundle over $\mathbf{R}^4$ and let $I \in \mathscr{A}(E)$ be a finite energy ASD instanton on E. Then the following holds.*

1. *If $a \in \ker \delta_I$ decays to zero at infinity, that is to say $\lim_{r\to\infty} \sup_{\partial B_r(0)} |a| = 0$, then $|\nabla^k a| = O(r^{-3-k})$ for $k \geq 0$.*
2. *If $(\xi, \omega) \in \ker \delta_I^*$ decays to zero at infinity, then $(\xi, \omega) = 0$.*

Proposition 5.4. *For $\ell + \delta \in (-3, -1)$, there is a constant $c > 0$ such that for all $t \in (-T', T')$, $\lambda \in (0, \Lambda]$, $\hat{\mathfrak{I}} \in \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ and $a \in \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}})$ the following holds*

$$\|\iota_{t,\lambda}\hat{\mathfrak{I}}\|_{C^{k,\alpha}_{\ell,\delta,t,\lambda}} \leq c\lambda^{-1-\ell}\|\hat{\mathfrak{I}}\|_{C^{k,\alpha}}$$
$$\|\pi_{t,\lambda}a\|_{C^{k,\alpha}} \leq c\lambda^{1+\ell}\|a\|_{C^{k,\alpha}_{\ell,\delta,t,\lambda}}.$$

In particular, the linear operators $\iota_{t,\lambda}\pi_{t,\lambda} \colon C^{k,\alpha}_{\ell,\delta,t,\lambda} \to C^{k,\alpha}_{\ell,\delta,t,\lambda}$ and $\rho_{t,\lambda} \colon C^{k,\alpha}_{\ell,\delta,t,\lambda} \to C^{k,\alpha}_{\ell,\delta,t,\lambda}$ are bounded uniformly in t and λ.

Proof. Let $\hat{\mathfrak{I}} \in \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$, then it follows from Proposition 5.3 and simple scaling considerations that

$$\|s_\lambda^* \hat{\mathfrak{I}}\|_{C^{k,\alpha}_{-3,0,t,\lambda}(V_{[0,\sigma),t})} \leq c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{k,\alpha}}$$

while at the same time

$$\|\chi_t^+\|_{C^{k,\alpha}_{3+\ell,\delta,t,\lambda}} \leq c\lambda^{-3-\ell}$$

since $\ell + \delta > -3$. This implies the first estimate by inequalities (5.1) and (5.2).

To prove the second inequality, note that for $\kappa \in (V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$

$$\begin{aligned}\int_{N_x P} \langle a, \chi_t^+ \kappa \rangle &\leq c \int_0^{\sqrt{\lambda}} \lambda^{2-\delta}(\lambda + r)^{\ell+\delta-3} r^3 \mathrm{d}r \cdot \|a\|_{L^\infty_{\ell,\delta,t,\lambda}} \|\kappa\|_{L^2} \\ &\quad + c \int_{\sqrt{\lambda}}^{\sigma} \lambda^2 r^{\ell-\delta}(\lambda + r)^{-3} r^3 \mathrm{d}r \cdot \|a\|_{L^\infty_{\ell,\delta,t,\lambda}} \|\kappa\|_{L^2} \\ &\leq c\lambda^{3+\ell} \|a\|_{L^\infty_{\ell,\delta,t,\lambda}} \|\kappa\|_{L^2}\end{aligned}$$

since $\ell + \delta < -1$. If κ is an element of an orthonormal basis of $(V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$ with respect to $\langle \iota_{t,\lambda} \cdot, \iota_{t,\lambda} \cdot \rangle$, then $\|\kappa\|_{L^2} \leq c\lambda$, since for $\kappa_1, \kappa_2 \in (V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$

$$\lambda^2 \langle \kappa_1, \kappa_2 \rangle_{L^2} \sim \langle \chi_t^+ s_\lambda^* \kappa_1, \chi_t^+ s_\lambda^* \kappa_2 \rangle_{L^2}$$

where $\sim$ means comparable uniformly in t and λ. Therefore

$$\|\pi_{t,\lambda} a\|_{L^\infty} \leq c\lambda^{1+\ell} \|a\|_{L^\infty_{\ell,\delta,t,\lambda}}$$

The estimates on the Hölder norms follow similarly. □

Now, for $\ell + \delta \in (-3, -1)$, we define $\mathfrak{A}^{k,\alpha}_{\ell,\delta,t,\lambda}$ to be the completion of $\mathfrak{A}_{t,\lambda}$ with respect to the $C^{k,\alpha}_{\ell,\delta,t,\lambda}$–norm. The spaces $C^{k,\alpha}_{\ell,\delta,t,\lambda}$, $\mathfrak{A}^{k,\alpha}_{\ell,\delta,t,\lambda}$ and $C^{k,\alpha}(P_t, \mathfrak{I}_t^* V\mathfrak{M}_t)$ form (trivial) bundles of topological vector spaces over $(-T', T') \times (0, \Lambda]$. We equip them with connections with respect to which $\iota_{t,\lambda}$, $\pi_{t,\lambda}$ and $\rho_{t,\lambda}$ are parallel. We also choose various other connections on bundles over $(-T', T') \times (0, \Lambda]$ whose fibres consists of twisted tensor fields over Y_t and NP_t compatible with each other and the above connections.

6 Estimate of the gluing error

In this section, as a first main step towards the proof of Theorem 1.1, we estimate the *gluing error*

$$\mathfrak{e}_{t,\lambda} := *(F_{A_{t,\lambda}} \wedge \psi_t) + \mu(t) \cdot \iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t.$$

Clearly, the gluing error $\mathfrak{e}_{t,\lambda}$ depends continuously differentiable on t and continuously on λ.

Proposition 6.1. *There is a constant $c_\mathfrak{e} > 0$ such that for $\delta \in (-1, 0)$, $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the following holds*

$$\begin{aligned}\|\rho_{t,\lambda} \mathfrak{e}_{t,\lambda}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} &\leq c_\mathfrak{e} \lambda^{2+\delta/2} & \|\rho_{t,\lambda} \partial_t \mathfrak{e}_{t,\lambda}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} &\leq c_\mathfrak{e} \lambda^{2+\delta/2} \\ \|\pi_{t,\lambda} \mathfrak{e}_{t,\lambda}\|_{C^{0,\alpha}} &\leq c_\mathfrak{e} \lambda & \|\pi_{t,\lambda} \partial_t \mathfrak{e}_{t,\lambda}\|_{C^{0,\alpha}} &\leq c_\mathfrak{e} \lambda.\end{aligned}$$

Here $\partial_t \mathfrak{e}_{t,\lambda}$ is to be understood with respect to the connection chosen at the end of Section 5. Before we can embark on the proof, we first need to obtain a better understanding of ψ_t near P_t.

Proposition 6.2. *Let P be an associative submanifold in a G_2–manifold (Y, ϕ). Then in a tubular neighbourhood of P, we can write the coassociative calibration $\psi = *\phi$ as*

$$\psi = \psi_0 + \psi_1 + \psi_{\geq 2}$$

where ψ_0 is defined as in equation (3.3), ψ_1 takes values in $\Lambda^2 T^*P \otimes \Lambda^+ N^*P$ and $\nabla^k \psi_1 = O\left(r^{1-k}\right)$ as well as $\nabla^k \psi_{\geq 2} = O\left(r^{2-k}\right)$ for $k = 0, 1$.

From Proposition 6.2 we obtain an expansion

$$\psi_t = \psi_{0,t} + \psi_{1,t} + \psi_{\geq 2,t}$$

with

$$\|\psi_{0,t}\|_{C^{0,\alpha}_{0,0,t,\lambda}} \leq c \qquad \|\psi_{1,t}\|_{C^{0,\alpha}_{1,0,t,\lambda}} \leq c \qquad \|\psi_{\geq 2,t}\|_{C^{0,\alpha}_{2,0,t,\lambda}} \leq c. \tag{6.3}$$

The terms of the expansion clearly depend continuously differentiable on t. Moreover, $\partial_t \psi_{0,t} = 0$, $\partial_t \psi_{1,t}$ still takes values in $\Lambda^2 T^* P_t \otimes \Lambda^+ N^* P_t$ and

$$\|\partial_t \psi_{1,t}\|_{C^{0,\alpha}_{1,0,t,\lambda}} \leq c \qquad \|\partial_t \psi_{\geq 2,t}\|_{C^{0,\alpha}_{2,0,t,\lambda}} \leq c. \tag{6.4}$$

Also note that we have

$$\| * - *_0 \|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{[0,\sigma),t})} \leq c \qquad \|\mathrm{vol}_{NP_t}\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{[0,\sigma),t})} \leq c/\lambda. \tag{6.5}$$

and similar estimates for the derivatives by t.

Proof of Proposition 6.2. If we pull the identity map of a tubular neighbourhood of P back to a tubular neighbourhood of the zero section of NP via the exponential map, then the Taylor expansion of its derivative around P can be expressed in the splitting $TNP = \pi^*TP \oplus \pi^*NP$ with $\pi \colon NP \to P$ being the canonical projection as

$$(x, y) \mapsto (x, y) + (\mathrm{II}_y(x), y) + O(|y|^2)$$

where II is the second fundamental form of P in Y, which we think of as a map from NP to $\mathrm{End}(TP)$. This immediately yields the desired expansion of ψ near P, since we know that ψ is given by ψ_0 along P. ∎

Proof of Proposition 6.1. We will show that

$$\|\mathfrak{e}_{t,\lambda}\|_{C^{0,\alpha}_{-2,0,t,\lambda}} = \| * (F_{A_{t,\lambda}} \wedge \psi_t) + \mu(t) \cdot \iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t \|_{C^{0,\alpha}_{-2,0,t,\lambda}} \leq c\lambda^2 \tag{6.6}$$

From this the estimates of $\mathfrak{e}_{t,\lambda}$ follow immediately by Proposition 5.4 and since $\|1\|_{C^{0,\alpha}_{0,\delta,t,\lambda}} \leq \lambda^{\delta/2}$. The estimates of $\partial_t \mathfrak{e}_{t,\lambda}$ follow by noticing that arguments used to estimate $\mathfrak{e}_{t,\lambda}$ can be differentiated by t since our whole setup depends smoothly on t. We leave it to the reader to fill in the details.

To prove the estimate (6.6) we proceed in four steps. The first step serves to estimate the contribution coming from $I_{t,\lambda}$. In the last three steps we estimate $\mathfrak{e}_{t,\lambda}$ separately on the subsets

$$V_{[0,\lambda),t} \qquad V_{[\lambda,\sigma/2),t} \qquad V_{[\sigma/2,\sigma),t}$$

constituting $V_{[0,\sigma),t}$ which contains the support of $\mathfrak{e}_{t,\lambda}$. We will prove that the $C^{0,\alpha}_{-2,0,t,\lambda}$–norm of $\mathfrak{e}_{t,\lambda}$ is bounded by $c\lambda^2$ on each of them. This immediately implies the above assertion.

Step 1. *We prove that*

$$\left\| * \left((F_{I_{t,\lambda}} - F_{A_t|_{P_t}}) \wedge \psi_t \right) + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_{t,\lambda} \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \leq c\lambda^2$$

It follows from Theorem 3.5 that in $V_{[0,\sigma),t}$

$$*_0 (F_{I_{t,\lambda}} \wedge (\psi_{0,t} - \mathrm{vol}_{NP_t})) + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_{t,\lambda} = 0.$$

By considerations of bi-degree and since $F^{0,2}_{I_{t,\lambda}}$ is anti-self-dual

$$F^{1,1}_{I_{t,\lambda}} \wedge \mathrm{vol}_{NP_t} = 0 \qquad F^{0,2}_{I_{t,\lambda}} \wedge \mathrm{vol}_{NP_t} = 0 \qquad F^{0,2}_{I_{t,\lambda}} \wedge \psi_{1,t} = 0.$$

Combining these two observations yields

$$\begin{aligned}
&\left\| * \left((F_{I_{t,\lambda}} - F_{A_t|_{P_t}}) \wedge \psi_t \right) + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_{t,\lambda} \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \\
&\leq \left\| (F_{I_{t,\lambda}} - F_{A_t|_{P_t}}) \wedge (\mathrm{vol}_{NP_t} + \psi_{1,t} + \psi_{\geq 2,t}) \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \\
&\quad + \left\| (* - *_0) \left((F_{I_{t,\lambda}} - F_{A_t|_{P_t}}) \wedge (\psi_{0,t} - \mathrm{vol}_{NP_t}) \right) \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \\
&\leq \left\| (F_{I_{t,\lambda}} - F_{A_t|P_t})^{2,0} \wedge (\mathrm{vol}_{NP_t} + \psi_{\geq 2,t}) \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \\
&\quad + \left\| F^{1,1}_{I_{t,\lambda}} \wedge (\psi_{1,t} + \psi_{\geq 2,t}) \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \\
&\quad + \left\| F^{0,2}_{I_{t,\lambda}} \wedge \psi_{\geq 2,t} \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \\
&\quad + \left\| (* - *_0) \left((F_{I_{t,\lambda}} - F_{A_t|_{P_t}})^{2,0} \wedge (\psi_{0,t} - \mathrm{vol}_{NP_t}) \right) \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})} \\
&\quad + \left\| (* - *_0) \left(F^{1,1}_{I_{t,\lambda}} \wedge \psi_{0,t} \right) \right\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})}
\end{aligned}$$

If follows from the construction of $I_{t,\lambda}$ and Proposition 5.3 that

$$\begin{gathered}
\| F^{2,0}_{I_{t,\lambda}} - F_{A_t|_{P_t}} \|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\sigma),t})} \leq c\lambda^3 \\
\| F^{1,1}_{I_{t,\lambda}} \|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\sigma),t})} \leq c\lambda^2 \qquad \| F^{0,2}_{I_{t,\lambda}} \|_{C^{0,\alpha}_{-4,0,t,\lambda}(V_{[0,\sigma),t})} \leq c\lambda^2.
\end{gathered}$$

Combining this with inequalities (6.3) and (6.5) yields the desired estimate.

Step 2. *Estimate of the gluing error on* $V_{[\lambda,\sigma/2),t}$.

In $V_{[0,\lambda),t}$ the curvature of $A_{t,\lambda}$ is given by

$$F_{A_{t,\lambda}} = F_{I_{t,\lambda}} + \chi^-_{t,\lambda} \mathrm{d}_{I_{t,\lambda}} a_t + \frac{1}{2} (\chi^-_{t,\lambda})^2 [a_t \wedge a_t] + \mathrm{d}\chi^-_{t,\lambda} \wedge a_t$$

By Step 1 and since

$$\| F_{A_t|_{P_t}} \wedge \psi_t \|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\lambda),t})} \leq \| 1 \|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\lambda),t})} \| F_{A_t|_{P_t}} \wedge \psi_t \|_{C^{0,\alpha}_{0,0,t,\lambda}(V_{[0,\lambda),t})} \leq c\lambda^2$$

it suffices to estimate $F_{A_{t,\lambda}} - F_{I_{t,\lambda}}$. The cut-off functions $\chi^-_{t,\lambda}$ were constructed so that

$$\| \chi^-_{t,\lambda} \|_{C^{1,\alpha}_{0,0,t,\lambda}} \leq c$$

and it follows from the fact that A_t is in radial gauge with respect to $A_t|_{P_t}$ and Proposition 5.3 that

$$\begin{aligned}\|i_{t,\lambda}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\sigma),t})} + \|\mathrm{d}_{I_{t,\lambda}} i_{t,\lambda}\|_{C^{0,\alpha}_{-4,0,t,\lambda}(V_{[0,\sigma),t)})} &\leq c\lambda^2 \\ \|a_t\|_{C^{0,\alpha}_{0,0,t,\lambda}(V_{(\lambda,\sigma),t})} + \|\mathrm{d}_{A_t|_{P_t}} a_t\|_{C^{0,\alpha}_{0,0,t,\lambda}(V_{(\lambda,\sigma),t})} &\leq c\end{aligned}$$

Hence,

$$\begin{aligned}&\|F_{A_{t,\lambda}} - F_{I_{t,\lambda}}\|_{C^{0,\alpha}_{-2,0,t,\lambda}} \\ &\quad\leq \|\chi^-_{t,\lambda}\|_{C^{1,\alpha}_{0,0,t,\lambda}(V_{[0,\lambda),t})} \cdot \|\mathrm{d}_{A_t|_{P_t}} a_t\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{(\sigma,\lambda),t})} \\ &\qquad + \|\chi^-_{t,\lambda}\|_{C^{1,\alpha}_{0,0,t,\lambda}(V_{[0,\lambda),t})} \|i_{t,\lambda}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\lambda),t})} \|a_t\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{(\sigma,\lambda),t})} \\ &\qquad + \frac{3}{2}\|\chi^-_{t,\lambda}\|^2_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\lambda),t})} \cdot \|a_t\|^2_{C^{0,\alpha}_{1,0,t,\lambda}(V_{(\sigma,\lambda),t})} \\ &\qquad + \|\mathrm{d}\chi^-_{t,\lambda}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\lambda),t})} \|a_t\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{(\sigma,\lambda),t})} \\ &\quad\leq c\lambda^2\end{aligned}$$

as desired.

Step 3. *Estimate of the gluing error on $V_{[\lambda,\sigma/2),t}$.*

Since in $V_{[\lambda,\sigma/2),t}$ the curvature of $A_{t,\lambda}$ is given by

$$F_{A_{t,\lambda}} = F_{A_t} + F_{I_{t,\lambda}} - F_{A_t|_{P_t}}$$

and $F_{A_t} \wedge \psi_t = 0$ the estimate follows at once from Step 1.

Step 4. *Estimate of the gluing error on $V_{[\sigma/2,\sigma),t}$.*

In $V_{[\sigma/2,\sigma),t}$ the curvature of $A_{t,\lambda}$ is given by

$$F_{A_{t,\lambda}} = F_{A_t} + \chi^+_t \mathrm{d}_{A_t} i_{t,\lambda} + \frac{1}{2}(\chi^+_t)^2[i_{t,\lambda} \wedge i_{t,\lambda}] + \mathrm{d}\chi^+_t \wedge i_{t,\lambda}.$$

Since $\|\chi^+_t\|_{C^{1,\alpha}_{2,0:t,\lambda}(V_{[\sigma/2,\sigma),t})} \leq c$ it follows that

$$\begin{aligned}&\|F_{A_{t,\lambda}} - F_{A_t}\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \\ &\quad\leq \|\chi^+_t\|_{C^{0,\alpha}_{2,0:t,\lambda}(V_{[\sigma/2,\sigma),t})} \|\mathrm{d}_{I_{t,\lambda}} i_{t,\lambda}\|_{C^{0,\alpha}_{-4,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \\ &\qquad + \|\chi^+_t\|_{C^{0,\alpha}_{0,0:t,\lambda}(V_{[\sigma/2,\sigma),t})} \|a_t\|_{C^{0,\alpha}_{1,0:t,\lambda}(V_{[\sigma/2,\sigma),t})} \|i_{t,\lambda}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \\ &\qquad + \frac{3}{2}\|\chi^+_t\|^2_{C^{0,\alpha}_{2,0:t,\lambda}(V_{[\sigma/2,\sigma),t})} \|i_{t,\lambda}\|^2_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \\ &\qquad + \|\mathrm{d}\chi^+_t\|_{C^{0,\alpha}_{1,0:t,\lambda}(V_{[\sigma/2,\sigma),t})} \|i_{t,\lambda}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \\ &\quad\leq c\lambda^2.\end{aligned}$$

Thus we are left with estimating

$$\begin{aligned}&\|\iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t - \hat{v} \circ \mathfrak{I}_{t,\lambda}\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \\ &\qquad\leq c\|\chi^+_t\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \cdot \|\hat{v} \circ \mathfrak{I}_{t,\lambda}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[\sigma/2,\sigma),t})}.\end{aligned}$$

But, since $\|\chi^+_t\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{[\sigma/2,\sigma),t})} \leq c$ and from Proposition 5.3 it follows that

$$\|\hat{v} \circ \mathfrak{I}_{t,\lambda}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\sigma),t})} \leq c\lambda^2$$

this term is bounded by $c\lambda^2$. This concludes the proof. ∎

7 Linear estimates

Proposition 7.1. *For $\delta \in (-1,0)$ there is a constant $c_{\mathfrak{R}} > 0$ and a family of linear operators $\mathfrak{R}_{t,\lambda}\colon \mathfrak{A}^{0,\alpha}_{-2,\delta,t,\lambda} \to \mathfrak{A}^{1,\alpha}_{-1,\delta,t,\lambda}$ depending continuously differentiable on $t \in (-T', T')$ and continuously on $\lambda \in (0, \Lambda]$ such that*

$$\mathfrak{L}_{t,\lambda}\mathfrak{R}_{t,\lambda} = \mathrm{id}_{\mathfrak{A}^{0,\alpha}_{-2,\delta,t,\lambda}}$$

and

$$\|\mathfrak{R}_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}} \leq c_{\mathfrak{R}}\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \qquad \|\partial_t\mathfrak{R}_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}} \leq c_{\mathfrak{R}}\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}$$

for all $t \in (-T', T')$, $\lambda \in (0, \Lambda]$ and $\underline{a} \in \mathfrak{A}^{0,\alpha}_{-2,\delta,t,\lambda}$.

The proof of this proposition proceeds by first studying local models for $\mathfrak{L}_{t,\lambda}$ near P_t in Section 7.1 as well as a model for $\mathfrak{L}_{t,\lambda}$ on the complement of P_t in Section 7.2 and then patching local approximate right inverses to obtain a global right inverse in Section 7.3.

Proposition 7.2. *There is a constant $c_{\mathfrak{S}} > 0$ and a family of linear operators $\mathfrak{S}_{t,\lambda}\colon C^{0,\alpha}(P_t, \mathfrak{I}_t^*V\mathfrak{M}_t) \to C^{1,\alpha}(P_t, \mathfrak{I}_t^*V\mathfrak{M}_t) \oplus \mathbf{R}$ depending continuously differentiable on $t \in (-T', T')$ and continuously on $\lambda \in (0, \Lambda]$ such that*

$$\hat{\mathfrak{K}}_{t,\lambda}\mathfrak{S}_{t,\lambda} = \mathrm{id}_{C^{0,\alpha}}$$

where $\hat{\mathfrak{K}}_{t,\lambda}\colon C^{1,\alpha}(P_t, \mathfrak{I}_t^*V\mathfrak{M}_t) \oplus \mathbf{R} \to C^{0,\alpha}(P_t, \mathfrak{I}_t^*V\mathfrak{M}_t)$ is defined by

$$\hat{\mathfrak{K}}_{t,\lambda}(\hat{\mathfrak{I}}, \eta) := \mathfrak{K}_{t,\lambda}\hat{\mathfrak{I}} + \eta\hat{v} \circ \mathfrak{I}_t$$

and

$$\|\mathfrak{S}_{t,\lambda}\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \leq c_{\mathfrak{S}}\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}} \qquad \|\partial_t\mathfrak{S}_{t,\lambda}\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \leq c_{\mathfrak{S}}\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}$$

for all $t \in (-T', T')$, $\lambda \in (0, \Lambda]$ and $\hat{\mathfrak{I}} \in C^{0,\alpha}(P_t, \mathfrak{I}_t^*V\mathfrak{M}_t)$.

This will turn out to be a consequence of the fact that $\hat{\mathfrak{K}}_{t,\lambda}$ is a small perturbation of the Fueter operator $F_{\mathfrak{I}_t}$.

Proposition 7.3. *For $\delta \in (-1,0)$ there are constants $c_{\mathfrak{p}}, c_{\mathfrak{q}} > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the following holds*

$$\|\mathfrak{p}_{t,\lambda}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,\delta,t\lambda}} \leq c_{\mathfrak{p}}\lambda^{2+\delta/2}\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \qquad \|\partial_t\mathfrak{p}_{t,\lambda}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,\delta,t\lambda}} \leq c_{\mathfrak{p}}\lambda^{2+\delta/2}\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}$$
$$\|\mathfrak{q}_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} \leq c_{\mathfrak{q}}\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t\lambda}} \qquad \|\partial_t\mathfrak{q}_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} \leq c_{\mathfrak{q}}\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t\lambda}}$$

for all $\underline{a} \in \mathfrak{A}^{1,\alpha}_{-1,\delta,t,\lambda}$ and $\hat{\mathfrak{I}} \in C^{1,\alpha}(P_t, \mathfrak{I}_t^*V\mathfrak{M}_t)$.

The reader may fear that the estimate on $\mathfrak{q}_{t,\lambda}$ is insufficient, because it does not become small as λ goes to zero. This apparent defect disappears upon realising that the appropriately scaled norm is really $\lambda\|\cdot\|_{C^{k,\alpha}}$.

7.1 Model on $\mathbf{R}^3 \times \mathbf{R}^4$

Pick a point a point $x \in P_t$. Let I be the ASD instanton obtained by restricting I_t to N_xP and let E be the restriction of $E(\mathfrak{I}_t)$ to N_xP. Keeping notation we pullback

E and I to T_xY. The is no loss in generality if we identify $T_xY = T_xP_t \oplus N_xP_t$ with $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$ in such a way that the coassociative calibration is identified with

$$\psi_0 := \frac{1}{2}\omega_1 \wedge \omega_1 + \mathrm{d}x^{12} \wedge \omega_3 + \mathrm{d}x^{23} \wedge \omega_1 + \mathrm{d}x^{31} \wedge \omega_2$$

where

$$\omega_1 := \mathrm{d}x^{45} + \mathrm{d}x^{67} \qquad \omega_1 := \mathrm{d}x^{46} - \mathrm{d}x^{47} \qquad \omega_1 := -\mathrm{d}x^{46} - \mathrm{d}x^{56}.$$

We define $L_I : \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E) \to \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E)$ by

$$L_I = \begin{pmatrix} 0 & \mathrm{d}_I^* \\ \mathrm{d}_I & *(\psi_0 \wedge \mathrm{d}_I) \end{pmatrix}.$$

With weight functions given by

$$w(x) := 1 + |x| \qquad w(x,y) := \min\{w(x), w(y)\}$$

we define weighted Hölder norms depending on a parameter $\beta \in \mathbf{R}$ for tensor fields f on $\mathbf{R}^7$ (with values in $\mathfrak{g}_E$) by

$$\tag{7.4} \begin{aligned} [f]_{C^{0,\alpha}_\beta} &:= \sup_{d(x,y) \le w(x,y)} w(x,y)^{\alpha-\beta} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \\ \|f\|_{L^\infty_\beta} &:= \|w^{-\beta} f\|_{L^\infty} \\ \|f\|_{C^{k,\alpha}_\beta} &:= \sum_{j=0}^k \|\nabla^j f\|_{L^\infty_{\beta-j}} + [\nabla^j f]_{C^{0,\alpha}_{\beta-j}}. \end{aligned}$$

Here the metrics are the ones induced by the standard metric on $\mathbf{R}^7$ and the metric on $\mathfrak{g}_E$ while the connection is the one induced by I. We use parallel transport with respect to this connection to compare $f(x)$ and $f(y)$ in the definition of the weighted Hölder semi-norm.

Fix two constants $\epsilon > 0$ and $N > 0$. Identify

$$V_{\epsilon,N,t,\lambda} := V_{[0,N\sqrt{\lambda}),t} \cap B_\epsilon(x)$$

with a small neighbourhood of zero in $\mathbf{R}^3 \times \mathbf{R}^4$ ($\cong T_xY$) using the exponential map. Since on $V_{\epsilon,N,t,\lambda}$ the bundle $E_{t,\lambda}$ is trivial, provided ϵ is sufficiently small, we can identify tensor fields f (with values in $\mathfrak{g}_{E_{t,\lambda}}$) with ($\mathfrak{g}$–valued) tensor fields f defined in a neighbourhood of zero in $\mathbf{R}^3 \oplus \mathbf{R}^4$. For a weight k, we define a rescaled ($\mathfrak{g}$–valued) tensor field $s_{k,\lambda} f$ on $U_{\epsilon,N,\lambda} := \{(x,y) \in \mathbf{R}^3 \oplus \mathbf{R}^4 : |x| < \epsilon, |y| \le N\sqrt{\lambda}\}$ by

$$(s_{k,\lambda} f)(x,y) := \lambda^k f(\lambda x, \lambda y).$$

Now it is easy to see the following result.

Proposition 7.5. *There is a constant $c > 0$ such that for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\|f\|_{C^{k,\alpha}_{\ell,\delta,t,\lambda}(V_{\epsilon,N,t,\lambda})} \sim \lambda^{-k-\ell} \|s_{k,\lambda} f\|_{C^{k,\alpha}_{\ell+\delta}(U_{\epsilon,N,\lambda})}$$

and

$$\left\| L_{t,\lambda}\underline{a} - s_{2,\lambda}^{-1} L_I s_{1,\lambda} \underline{a} \right\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{\epsilon,N,t,\lambda})} \le c(\epsilon + N\sqrt{\lambda})\, \|\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\delta}(V_{\epsilon,N,t,\lambda})}.$$

To better understand L_I it is helpful to split it into

$$L_I = F + D_I$$

where F contains all the derivatives in the $\mathbf{R}^3$–direction, while D_I contains those in the $\mathbf{R}^4$–direction. If we identify $(\mathbf{R}^3)^*$ with Λ^+ via $\mathrm{d}x^i \mapsto \omega_i$, then F is given by

$$F(\xi, \omega, a) = \sum_{i=1}^{3} \left(-\langle \partial_i \omega, \omega_i \rangle, \partial_i \xi \cdot \omega_i, *_4(\partial_i a \wedge \omega_i)\right)$$

and D_I by

$$D_I = \begin{pmatrix} 0 & \delta_I \\ \delta_I^* & 0 \end{pmatrix}$$

where $\delta_I \colon \Omega^1(\mathbf{R}^4, \mathfrak{g}_E) \to \Omega^0(\mathbf{R}^4, \mathfrak{g}_E) \oplus \Omega^+(\mathbf{R}^4, \mathfrak{g}_E)$ is the linear operator associated to I defined by equation (3.1). It is an easy exercise to verify that $F^*F = \Delta_{\mathbf{R}^3}$ and $D_I^* F + F^* D_I = 0$. Hence

$$L_I^* L_I = \Delta_{\mathbf{R}^3} + \begin{pmatrix} \delta_I \delta_I^* & \\ & \delta_I^* \delta_I \end{pmatrix}$$

Lemma 7.6. *Let E be a vector bundle of bounded geometry over a Riemannian manifold X of bounded geometry and suppose that $D \colon C^\infty(X, E) \to C^\infty(X, E)$ is a bounded uniformly elliptic operator of second order which is positive, i.e., $\langle Da, a \rangle \geq 0$ for all $a \in W^{2,2}(X, E)$, and formally self-adjoint. If $a \in C^\infty(\mathbf{R}^n \times X, E)$ satisfies*

$$(\Delta_{\mathbf{R}^n} + D)a = 0$$

and there are $s \in \mathbf{R}$ and $p \in (1, \infty)$ such that $\|a(x, \cdot)\|_{W^{s,p}}$ is bounded independent of $x \in \mathbf{R}^n$, then a is constant in the $\mathbf{R}^n$–direction, that is $a(x, y) = a(y)$.

Proof. See [Wal11, Appendix A]. □

An immediate consequence of this lemma and Proposition 5.3 is that for $\beta < 0$ the kernel of $L_I \colon C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ can be identified with the L^2–kernel of δ_I. But also note that if $\beta \in (-2, 0)$, κ is in the kernel of δ_I and χ is a smooth function on $\mathbf{R}^3$, then

$$\|L_I(\chi\kappa)\|_{C^{0,\alpha}_{\beta-1}} \leq c\|\mathrm{d}\chi\|_{C^{0,\alpha}}$$

which can be made arbitrarily small by stretching the support of χ while keeping the value of $\|\chi\kappa\|_{C^{0,\alpha}_\beta}$ essentially fixed. Therefore, the linear operator $L_I \colon C^{1,\alpha}_\beta \to C^{0,\alpha}_{\beta-1}$ can not be Fredholm, let alone invertible. To overcome this issue, for $\beta < -1$, we define $\pi_I \colon C^{k,\alpha}_\beta \to C^{k,\alpha}(\mathbf{R}^3, \ker \delta_I)$ by

$$\pi_I(x) := \sum_\kappa \langle a(x, \cdot), \kappa \rangle_{L^2(\mathbf{R}^4)} \kappa$$

where κ runs through an L^2 orthonormal basis for δ_I and set

$$\mathfrak{A}^{k,\alpha}_\beta := \ker \pi_I \cap C^{k,\alpha}_\beta.$$

Now it follows from the discussion at the beginning of Section 3, that F and D_I commute with π_I. Hence, L_I defines a linear operator $L_I \colon \mathfrak{A}^{1,\alpha}_\beta \to \mathfrak{A}^{0,\alpha}_{\beta-1}$.

Proposition 7.7. *For $\beta \in (-2, -1)$ the linear operator $L_I \colon \mathfrak{A}^{1,\alpha}_\beta \to \mathfrak{A}^{0,\alpha}_{\beta-1}$ is invertible.*

The proof relies on the following estimate.

Proposition 7.8. *For $\beta \in (-3,-1)$ there is a constant $c > 0$ such that for all $\underline{a} \in \mathfrak{A}^{1,\alpha}_{\beta}$ the following holds*

$$\|\underline{a}\|_{C^{1,\alpha}_{\beta}} \leq c\|L_I\underline{a}\|_{C^{0,\alpha}_{\beta-1}}.$$

Proof of Proposition 7.7. From Proposition 7.8 it follows that $L_I\colon \mathfrak{A}^{1,\alpha}_{\beta} \to \mathfrak{A}^{0,\alpha}_{\beta-1}$ is injective and its image is closed. Thus we can identify its cokernel with the kernel of $L_I^*\colon (\mathfrak{A}^{0,\alpha}_{\beta-1})^* \to (\mathfrak{A}^{1,\alpha}_{\beta})^*$. Since $\beta \in (-2,-1)$, the image of π_I is contained in $\mathfrak{A}^{0,\alpha}_{\beta-1}$ and thus have $C^{0,\alpha}_{\beta-1} = \ker \pi_I \oplus \operatorname{im} \pi_I$. Via this splitting we can extend any $\underline{b} \in \ker L_I^*$ to an element of $(C^{0,\alpha}_{\beta-1})^*$ which still satisfies $L_I^*\underline{b} = 0$. By elliptic regularity $\underline{b}$ is smooth and it follows from Lemma 7.6 that $\underline{b}$ is invariant under translations in the $\mathbf{R}^3$–direction. Now $\underline{b}$ must be contained in $C^{1,\alpha}_{-3-\beta}$. Since L_I is formally self-adjoint we have $L_I\underline{b} = 0$. But then it follows that $\underline{b} = 0$, since $-3-\beta \in (-3,-1)$. □

The proof of Proposition 7.8 is very similar to that of Proposition 8.2 in [Wal11]. Nevertheless we provide the details here for the readers convenience. We begin with the following Schauder estimate.

Proposition 7.9. *For $\beta \in \mathbf{R}$ there is a constant $c > 0$ such that the following estimate holds*

$$\|\underline{a}\|_{C^{1,\alpha}_{\beta}} \leq c\left(\|L_I\underline{a}\|_{C^{0,\alpha}_{\beta-1}} + \|\underline{a}\|_{L^\infty_\beta}\right).$$

Proof. The desired estimate is local in the sense that is enough to prove estimates of the form

$$\|\underline{a}\|_{C^{1,\alpha}_{\beta}(U_i)} \leq c(\|L_I\underline{a}\|_{C^{0,\alpha}_{\beta-1}} + \|\underline{a}\|_{L^\infty_\beta})$$

with $c > 0$ independent of i, where $\{U_i\}$ is a suitable open cover of $\mathbf{R}^3 \times \mathbf{R}^4$.

Fix $R > 0$ suitably large and set $U_0 = \{(x,y) \in \mathbf{R}^3 \times \mathbf{R}^4 : |y| \leq R\}$. Then there clearly is a constant $c > 0$ such that the above estimate holds for $U_i = U_0$. Pick a sequence $(x_i, y_i) \in \mathbf{R}^3 \times \mathbf{R}^4$ such that $r_i := |y_i| \geq R$ and the balls $U_i := B_{r_i/8}(x_i,y_i)$ cover the complement of U_0. On U_i we have a Schauder estimate of the form

$$\begin{aligned}\|\underline{a}\|_{L^\infty(U_i)} &+ r_i^\alpha[\underline{a}]_{C^{0,\alpha}(U_i)} + r_i\|\nabla_A\underline{a}\|_{L^\infty(U_i)} + r_i^{1+\alpha}[\nabla_A\underline{a}]_{C^{0,\alpha}(U_i)} \\ &\leq c\left(r_i\|L_I\underline{a}\|_{L^\infty(V_i)} + r_i^{1+\alpha}[L_I\underline{a}]_{C^{0,\alpha}(V_i)} + \|\underline{a}\|_{L^\infty(V_i)}\right)\end{aligned}$$

where $V_i = B_{r_i/4}(x_i,y_i)$. The constant $c > 0$ depends continuously on the coefficients of L_I over V_i and it is thus easy to see that one can find a constant $c > 0$ such that the above estimate holds for all i. Since on V_i we have $\frac{1}{2}r_i \leq w \leq 2r_i$ multiplying the above Schauder estimate by $r_i^{-\beta}$ yields the desired local estimate. □

In view of the above Schauder estimate Proposition 7.8 is an immediate consequence of the following result.

Proposition 7.10. *For $\beta \in (-3,1)$ there is a constant $c > 0$ such that the following holds*

$$\|\underline{a}\|_{L^\infty_\beta} \leq c\|L_I\underline{a}\|_{C^{0,\alpha}_{\beta-1}}$$

for all $\underline{a} \in \mathfrak{A}^{1,\alpha}_{\beta}$.

Proof. Suppose the estimate does not hold. Then there exists a sequence $\underline{a}_i \in \mathfrak{A}^{1,\alpha}_\beta$. and such that

$$\|\underline{a}_i\|_{L^\infty_\beta} = 1 \qquad \|L_I \underline{a}_i\|_{C^{0,\alpha}_{\beta-1}} \leq \frac{1}{i}.$$

Hence by the above Schauder estimate

$$\|\underline{a}_i\|_{C^{1,\alpha}_\beta} \leq 2c$$

Pick $(x_i, y_i) \in \mathbf{R}^3 \oplus \mathbf{R}^4$ such that

$$w(x_i, y_i)^{-\beta} |\underline{a}_i(x_i, y_i)| = 1.$$

By translation we can assume that $x_i = 0$. Then with out loss of generality one of the following two cases must occur. We rule out both of them thus proving the estimate.

Case 1. *The sequence $|y_i|$ stays bounded.*

Let K be a compact subset of $\mathbf{R}^7$. When restricted to K, the elements $\underline{a}_i$ are uniformly bounded in $C^{1,\alpha}$. Thus, by Arzelà-Ascoli, we can assume (after passing to a subsequence) that $\underline{a}_i$ converges to a limit $\underline{a}$ in $C^{1,\alpha/2}$. Since K was arbitrary this yields $\underline{a} \in \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E)$ satisfying

$$|\underline{a}|(x,y) < c|y|^\beta \qquad |\nabla_I \underline{a}|(x,y) < c|y|^{\beta-1}$$

as well as

$$L_I \underline{a} = 0 \qquad \pi_I \underline{a} = 0$$

It follows from Lemma 7.6 that $\underline{a} = 0$. But we can assume that y_i converges to some point $y \in \mathbf{R}^4$ for which we would have $|\underline{a}|(0,y) = w(0,y)^\beta \neq 0$. This is a contradiction.

Case 2. *The sequence $|y_i|$ goes to infinity.*

Define a rescaled sequence $\underline{\tilde{a}}_i$ by

$$\underline{\tilde{a}}_i(x,y) := |y_i|^{1-\delta}(\xi_i, a_i)(|y_i|x, |y_i|y)$$

and set $\tilde{y}_i = y_i/|y_i|$. Moreover, we introduce weighted Hölder norms $\|\cdot\|_{\tilde{C}^{k,\alpha}_\beta}$ for twisted tensor field over $\mathbf{R}^3 \times \left(\mathbf{R}^4 \setminus \{0\}\right)$ which are defined as in equation (7.4) but with weight function $\tilde{w}(x,y) = |y|$ instead of $w(x,y) = 1 + |y|$. The rescaled sequence then satisfies

$$\|\underline{\tilde{a}}_i\|_{\tilde{C}^{1,\alpha}_\beta} \leq 2c \qquad \|L\underline{\tilde{a}}_i\|_{\tilde{C}^{0,\alpha}_{\beta-1}} \leq 2/i \qquad \tilde{w}(0,\tilde{y}_i)^{-\beta} |\underline{\tilde{a}}_i(0,\tilde{y}_i)| \geq 1/2$$

where L is defined by

$$L\underline{a} := (\mathrm{d}^* a, \mathrm{d}\xi + *(\psi \wedge \mathrm{d}a))$$

with $\underline{a} = (\xi, a)$.

We can now pass to a limit using Arzelà-Ascoli as before to obtain $\underline{\tilde{a}}$ defined over $\mathbf{R}^3 \times \left(\mathbf{R}^4 \setminus \{0\}\right)$ satisfying

$$|\underline{\tilde{a}}|(x,y) < c|y|^\beta \qquad |\nabla \underline{\tilde{a}}| < c|y|^{\beta-1} \qquad L\underline{\tilde{a}} = 0.$$

Since $\beta > -3$, $L\underline{\tilde{a}} = 0$ holds on all of $\mathbf{R}^7$ in the sense of distributions. Hence, by elliptic regularity, $\underline{\tilde{a}}$ extends to a smooth solution of $L\underline{\tilde{a}} = 0$ on $\mathbf{R}^7$. Since $L^*L = \Delta_{\mathbf{R}^3} + \Delta_{\mathbf{R}^4}$, it follows from Lemma 7.6 that $\underline{\tilde{a}}$ is invariant in the $\mathbf{R}^3$–direction. Therefore we can think of the components of $\underline{\tilde{a}}$ as harmonic functions on $\mathbf{R}^4$. Since they decay to zero at infinity, they must vanish identically. On the other hand we know that $|\tilde{y}_i| = 1$ and thus (without loss of generality) $\tilde{y}_i$ converges some point $\tilde{y}$ in the unit sphere for which $|\underline{\tilde{a}}|(0,\tilde{y}) \geq \frac{1}{2}$, contradicting $\underline{\tilde{a}} = 0$. $\square$

7.2 Model away from P_t

Denote by $L_t := L_{A_t}$ the deformation operator associated to the G_2–instanton A_t. We define weighted Hölder norms $\|\cdot\|_{C^{k,\alpha}_{\beta,t}}$ for tensor fields (with values in $\mathfrak{g}_{E_t}$ on $Y \setminus P_t$ as in equation (7.4) but with weight function given by

$$w_t(x) := r_t(x) \qquad w_t(x,y) := \min\{w_t(x), w_t(y)\}.$$

If we fix a constant $N > 0$, then over $V_{[\sqrt{\lambda}/N,\infty),t}$ we can view a tensor field f with values in $\mathfrak{g}_{E_{t,\lambda}}$ as one which takes values in $\mathfrak{g}_{E_t}$ and vice versa.

Proposition 7.11. *There is a constant $c > 0$ such that for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ with respect to the above identification we have*

$$\|\underline{a}\|_{C^{k,\alpha}_{\ell,\delta,t,\lambda}(V_{[\sqrt{\lambda}/N,\infty),t})} \sim \|\underline{a}\|_{C^{k,\alpha}_{-\ell+\delta,t}(V_{[\sqrt{\lambda}/N,\infty),t})}$$

and

$$\|L_{t,\lambda}\underline{a} - L_t\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{[\sqrt{\lambda}/N,\infty),t})} \le c\sqrt{\lambda}/N \|\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}(V_{[\sqrt{\lambda}/N,\infty),t})}.$$

Proposition 7.12. *For $\beta \in (-3, 0)$ the operator $L_t \colon C^{1,\alpha}_{\beta,t} \to C^{0,\alpha}_{\beta-1,t}$ is invertible.*

Proof. The crucial ingredient is again the existence of an estimate of the form

$$\|\underline{a}\|_{C^{1,\alpha}_{\beta,t}} \le c\|L_t\underline{a}\|_{C^{0,\alpha}_{\beta-1,t}}$$

for $\beta \in (-3, 0)$. Assuming such an estimate it follows immediately that L_A is injective and its image is closed. Let $\underline{b} \in \ker L_t^* \cong \operatorname{coker} L_t$. Then using elliptic regularity and the fact that L_t is formally self-adjoint it can be seen that $\underline{b}$ represents an element in the kernel of $L_t \colon C^{1,\alpha}_{-3-\beta} \to C^{0,\alpha}_{-4-\beta}$. But then $\underline{b} = 0$ by the above estimate.

Now we are left with proving the above estimate. The argument is very similar to that used in the proof of Proposition 7.8. We only provide a brief sketch. The reader should have no trouble filling in the details. First of all we have the following Schauder estimate

$$\|\underline{a}\|_{C^{1,\alpha}_{\beta,t}} \le c(\|L_t\underline{a}\|_{C^{0,\alpha}_{\beta-1,t}} + \|\underline{a}\|_{L^\infty_{\beta,t}}).$$

To prove that

$$\|\underline{a}\|_{L^\infty_{\beta,t}} \le c\|L_t\underline{a}\|_{C^{0,\alpha}_{\beta-1,t}}$$

one argues by contradiction. If $\underline{a}_i$ is a sequence of counterexamples as before, then we can assume that it either gives rise to a non-trivial element in the kernel of $L_t \colon C^{1,\alpha}_{\beta} \to C^{0,\alpha}_{\beta-1}$ or localises in smaller and smaller neighbourhoods of P_t. To see that the first case cannot occur observe that if $\underline{a} \in C^{1,\alpha}_{\beta,t}$ solves L_t on $Y \setminus P_t$, then it follows that $L_t\underline{a} = 0$ on all of Y_t in the sense of distributions and thus $\underline{a}$ extends smoothly to Y_t, since $\beta > -3$. But A_t is unobstructed, hence L_t has trivial kernel and thus $\underline{a} = 0$. Thus we must be in the second case. Rescaling $\underline{a}_i$ near P_t as before yields a non-trivial harmonic function on $\mathbf{R}^3 \times \mathbf{R}^4 \setminus \{0\}$ which is bounded by a constant multiple of $|y|^\beta$. Since $\beta > -3$ the function extends to $\mathbf{R}^7$ and by Lemma 7.6 it is in variant in the $\mathbf{R}^3$–direction. But then it corresponds to a decaying harmonic function on $\mathbf{R}^4$ and hence must vanish identically. So the second case does not occur as well, proving that the claimed estimate must hold. ∎

7.3 Patching local inverses for $\mathfrak{L}_{t,\lambda}$

Proof of Proposition 7.1. We proceed in five steps. The reader should note that in the following it can be arranged that the "generic constant" $c > 0$ is not just independent of t and λ but also does not depend on any of the auxiliary choices we make.

Step 1. *We split $\underline{a} \in \mathfrak{A}^{0,\alpha}_{-2,\delta,t,\lambda}$ into parts localised on small balls centred in P_t and in the complement of P_t.*

With χ as in Section 4 we define $\chi_{t,\lambda} \colon Y_t \to [0,1]$ by

$$\chi_{t,\lambda}(x) := \chi(r_t(x)/\sqrt{\lambda}).$$

Then $\|\chi_{t,\lambda}\|_{C^{0,\alpha}_{0,0,t,\lambda}} \leq c$. Fix a small constant $\epsilon \geq \sqrt{\lambda}$. Then we can pick a finite number of points $\{x_\gamma : \gamma \in \Gamma\} \subset P_t$ such that the balls $B_\epsilon(x_\gamma)$ cover all of P_t and a partition of unity $1 = \sum_{\gamma\in\Gamma} \chi_\gamma$ subordinate to this cover such that

$$\|\chi_\gamma\|_{C^{0,\alpha}_{0,0,t,\lambda}(\mathrm{supp}(1-\chi_{t,\lambda}))} \leq c.$$

Now, we can write $\underline{a} \in \mathfrak{A}^{0,\alpha}_{-2,\delta,t,\lambda}$ as

$$\underline{a} = \sum_{\gamma\in\Gamma} \underline{a}_\gamma + \underline{a}_0$$

where

$$\underline{a}_\gamma := (1-\chi_\gamma)\chi_{t,\lambda}\underline{a} \qquad \underline{a}_0 := \chi_{t,\lambda}\underline{a}.$$

By construction we have

$$\sum_\gamma \|\underline{a}_\gamma\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \leq c\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \qquad \|\underline{a}_0\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \leq c\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}.$$

Step 2. *Construction of local approximate inverses.*

Fix a large constant $N > 1$. Let I_γ be an ASD instanton obtained by restricting I_t to $N_{x_\gamma} P_t$. Using the identifications and the notation of Section 7.1 we define

$$\underline{b}_\gamma := s_{1,\lambda}^{-1} L_{I_\gamma}^{-1} \rho_{I_\gamma} s_{2,\lambda} \underline{a}_\gamma$$

where $\rho_{I_\gamma} = \mathrm{id} - \pi_{I_\gamma}$. Since $\pi_{t,\lambda}\underline{a} = 0$ we have $\|\pi_I s_{2,\lambda}\underline{a}_\gamma\|_{C^{0,\alpha}_{-2+\delta}} \leq c\epsilon\|s_{2,\lambda}\underline{a}_\gamma\|_{C^{0,\alpha}_{-2+\delta}}$ and hence it follows from Proposition 7.5 that

$$\|\underline{b}_\gamma\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}(V_{\epsilon,N,t,\lambda})} \leq c\|\underline{a}_\gamma\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}}$$

as well as

$$\|L_{t,\lambda}\underline{b}_\gamma - \underline{a}_\gamma\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{\epsilon,N,t,\lambda})} \leq c(\epsilon + N\sqrt{\lambda})\|\underline{a}_\gamma\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}.$$

From Proposition 7.11 it follows that

$$\underline{b}_0 := L_t^{-1}\underline{a}_0$$

satisfies

$$\|\underline{b}_0\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{[\sqrt{\lambda}/N,\infty),t})} \leq c\|\underline{a}_0\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}}$$

and

$$\|L_{t,\lambda}\underline{b}_0 - \underline{a}_0\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{[\sqrt{\lambda}/N,\infty),t})} \leq c\sqrt{\lambda}\|\underline{a}_0\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}}$$

Step 3. *Patching local approximate inverses.*

To extend the local approximate inverses $\underline{b}_\gamma$ and $\underline{b}_0$ to all of Y we need to introduce additional cut-off functions. We choose $\beta^\pm_{t,\lambda,N}\colon Y \to [0,1]$ such that

$$\beta^-_{t,\lambda,N}(x) = \begin{cases} 1 & r_t(x) \le 2\sqrt{\lambda} \\ 0 & r_t(x) \ge 2N\sqrt{\lambda} \end{cases} \qquad \beta^+_{t,\lambda,N}(x) = \begin{cases} 0 & r_t(x) \le \sqrt{\lambda}/N \\ 1 & r_t(x) \ge \sqrt{\lambda} \end{cases}$$

and

$$\|\mathrm{d}\beta^\pm_{t,\lambda,N}\|_{C^{0,\alpha}_{-1,0,t,\lambda}} \le c/\log(N).$$

This can be arranged by interpolating between 0 and 1 logarithmically, i.e., by defining $\beta^-_{t,\lambda,N}$ as an appropriate smoothing of $\log(2N\sqrt{\lambda}/r_t)/\log(N)$ in the intermediate region and similarly $\beta^+_{t,\lambda,N}$ as a smoothing of $\log(Nr_t/\sqrt{\lambda})/\log(N)$. Moreover, we choose $\tilde\chi_\gamma\colon P_t \to [0,1]$ such that $\tilde\chi_\gamma$ equals one on $B_\epsilon(x_\gamma)$, $\tilde\chi_\gamma$ vanishes outside $B_{2\epsilon}(x_\gamma)$ and

$$\|\nabla\tilde\chi_\gamma\|_{C^{1,\alpha}_{0,0,t,\lambda}(\operatorname{supp}\beta^-_{t,\lambda,N})} \le c.$$

Now we define an approximate right inverse for $\underline{a}$ by

$$\tilde{\mathfrak{R}}_{t,\lambda}\underline{a} := \sum_\gamma \rho_{t,\lambda}\beta^-_{t,\lambda,N}\tilde\chi_\gamma\underline{b}_\gamma + \rho_{t,\lambda}\beta^+_{t,\lambda,N}\underline{b}_0.$$

From the above estimates and the ones in Steps 1 and 2 it follows that

$$\|\tilde{\mathfrak{R}}_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}} \le c\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}.$$

Clearly, $\tilde{\mathfrak{R}}_{t,\lambda}$ depends continuously differentiable on t and continuously on λ. It is not difficult to see that

$$\|\partial_t\tilde{\mathfrak{R}}_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}} \le c\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}.$$

Step 4. *Estimate of* $\mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda} - \mathrm{id}_{\mathfrak{A}}$.

Now to see that $\tilde{\mathfrak{R}}_{t,\lambda}$ is indeed an approximate inverse for $\mathfrak{L}_{t,\lambda} = \rho_{t,\lambda}L_{t,\lambda}$ we estimate as follows

$$\begin{aligned}\|\mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda}\underline{a} - \underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le{}& \left\|\mathfrak{L}_{t,\lambda}\pi_{t,\lambda}\left(\sum_\gamma \rho_{t,\lambda}\beta^-_{t,\lambda,N}\tilde\chi_i\underline{b}_i + \rho_{t,\lambda}\beta^+_{t,\lambda,N}\underline{b}_0\right)\right\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \\ &+ c\sum_\gamma \|L_{t,\lambda}\beta^-_{t,\lambda,N}\tilde\chi_\gamma\underline{b}_\gamma - \underline{a}_\gamma\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \\ &+ c\|L_{t,\lambda}\beta^+_{t,\lambda,N}\underline{b}_0 - \underline{a}_0\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}.\end{aligned}$$

The first term can be estimated by $c\lambda^{2+\delta/2}\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}$ using Proposition 7.3 and estimates from Steps 1 and 2. To estimate the second term note that $\beta^+_{t,\lambda,N}\tilde\chi_\gamma$ is equal to one on the support of $\underline{a}_\gamma$. Hence the second term is controlled by

$$\begin{aligned}c(\epsilon + N\sqrt{\lambda})\sum_\gamma \|\underline{a}_\gamma\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} &+ c\|\mathrm{d}\beta^-_{t,\lambda,N}\|_{C^{0,\alpha}_{-1,0,t,\lambda}} \cdot \|\underline{b}_\gamma\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}} \\ &+ c\|\mathrm{d}\tilde\chi_\gamma\|_{C^{0,\alpha}_{-1,0,t,\lambda}(\operatorname{supp}\beta^-_{t,\lambda,N})} \cdot \|\underline{b}_\gamma\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}} \\ &\le c\left(\epsilon + N\sqrt{\lambda} + 1/\log(N)\right)\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}\end{aligned}$$

Similarly, since $\beta^+_{t,\lambda,N} = 1$ on the support of $\underline{a}_0$ we can estimate the third term by

$$c(\sqrt{\lambda} + 1/\log(N))\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}.$$

Putting all these estimates together yields

$$\|\mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda}\underline{a} - \underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le c(\lambda^{2+\delta/2} + \epsilon + N\lambda + 1/\log(N))\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}.$$

By choosing ϵ sufficiently small and N sufficiently large we can arrange that

$$\|\mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda}\underline{a} - \underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le \frac{1}{2}\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}. \tag{7.13}$$

for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$. In a similar fashion one can estimate $\partial_t(\mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda})$ and arrange that

$$\|\partial_t(\mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda})\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le \frac{1}{2}\|\underline{a}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}. \tag{7.14}$$

Step 5. *Construction of $\mathfrak{R}_{t,\lambda}$.*

By the estimate (7.13) the series

$$\mathfrak{R}_{t,\lambda} := \tilde{\mathfrak{R}}_{t,\lambda}(\mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda})^{-1} = \tilde{\mathfrak{R}}_{t,\lambda}\sum_{k=0}^{\infty}\left(\mathrm{id}_{\mathfrak{A}} - \mathfrak{L}_{t,\lambda}\tilde{\mathfrak{R}}_{t,\lambda}\right)^k$$

converges and constitutes a right inverse for $\mathfrak{L}_{t,\lambda}$. Clearly, $\mathfrak{R}_{t,\lambda}$ is bounded uniformly in t and λ. It is also clear that $\mathfrak{R}_{t,\lambda}$ depends continuously on λ. To see that $\mathfrak{R}_{t,\lambda}$ depends continuously differentiable on t and that $\partial_t\mathfrak{R}_{t,\lambda}$ is bounded uniformly in t and λ, note that the (formal) derivative of the above series converges as well, because of inequality (7.14). This concludes the proof of Proposition 7.1. $\square$

7.4 Comparison of $\mathfrak{K}_{t,\lambda}$ with $F_{\mathfrak{I}_t}$

Proposition 7.15. *There is a constant $c > 0$ such that the following holds*

$$\|L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}} - \iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le c\lambda^{2+\delta/2}\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}$$
$$\|\partial_t(L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}} - \iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}})\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le c\lambda^{2+\delta/2}\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.$$

Proof. As in the proof of Proposition 6.1 it suffices to estimate the $C^{0,\alpha}_{-2,0,t,\lambda}$–norm. If we view $\Gamma(\mathfrak{I}_t^*V\mathfrak{M}_t)$ as a subspace of $\Omega^1(NP, \mathfrak{g}_E)$, then on this subspace the linearised Fueter operator $F_{\mathfrak{I}_t}$ agrees with

$$\tilde{L}_{t,\lambda}\underline{a} := \left(\mathrm{d}^*_{I_{t,\lambda}}a, *_0(\psi_{0,t}\wedge \mathrm{d}_{I_t}a) + \mathrm{d}_{I_{t,\lambda}}a\right)$$

because if $\hat{\mathfrak{I}} \in \Gamma(\mathfrak{I}_t^*V\mathfrak{M}_t)$, then $\hat{\mathfrak{I}}(x) \in \ker\delta_{I_t|_{N_xP_t}}$. Therefore, we have

$$\begin{aligned}\|L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}} - \iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,0,t,\lambda}} \le{}& \|L_{I_{t,\lambda}}(\iota_{t,\lambda}\hat{\mathfrak{I}} - \hat{\mathfrak{I}})\|_{C^{0,\alpha}_{-1,0,t,\lambda}(V_{[0,\sigma),t})}\\ &+ \|(L_{t,\lambda} - L_{I_{t,\lambda}})\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,0,t,\lambda}(V_{[0,\sigma),t})}\\ &+ \|\iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}} - F_{\mathfrak{I}_t}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-1,0,t,\lambda}(V_{[0,\sigma),t})}\end{aligned}$$

The first and last term are supported in $V_{[\sigma/2,\sigma),t}$ and are easily bounded by $c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}$ using an argument similar to the one in Step 4 in the proof of Proposition 6.1. We can estimate the second term by

$$\begin{aligned}\|(L_{t,\lambda}-\tilde{L}_{t,\lambda})\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{[0,\sigma),t})} &\leq \|\psi_t\wedge[(A_{t,\lambda}-I_{t,\lambda})\wedge\hat{\mathfrak{I}}]\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{[0,\sigma),t})}\\ &\quad+\|(\psi_{1,t}+\psi_{\geq 2,t})\wedge \mathrm{d}_{I_{t,\lambda}}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{[0,\sigma),t})}\\ &\quad+\|(*-*_0)(\psi_0\wedge \mathrm{d}_{I_{t,\lambda}}\hat{\mathfrak{I}})\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}(V_{[0,\sigma),t})}\end{aligned}$$

Since, by Proposition 5.3,

$$\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\sigma),t})}\leq c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}$$

and, moreover,

$$\|A_{t,\lambda}-I_{t,\lambda}\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{[0,\sigma),t})}=\|\chi^-_{t,\lambda}a_t+(\chi^+_t-1)i_{t,\lambda}\|_{C^{0,\alpha}_{1,0,t,\lambda}(V_{[0,\sigma),t})}\leq c \tag{7.16}$$

the first term is bounded by $c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}$. To estimate the last two terms, note that it follows from Proposition 5.3 that

$$\|(\mathrm{d}_{I_{t,\lambda}}\hat{\mathfrak{I}})^{1,1}\|_{C^{0,\alpha}_{-3,0,t,\lambda}(V_{[0,\sigma),t})}+\|(\mathrm{d}_{I_{t,\lambda}}\hat{\mathfrak{I}})^{0,2}\|_{C^{0,\alpha}_{-4,0,t,\lambda}(V_{[0,\sigma),t})}\leq c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}$$

Since $\hat{\mathfrak{I}}(x)\in\ker\delta_{I_t|N_xP_t}$ we also have

$$\psi_0\wedge(\mathrm{d}_{I_{t,\lambda}}\hat{\mathfrak{I}})^{0,2}=0.$$

Combining these observations with inequalities (6.3) and (6.5) yields the desired estimate of the last two terms. This concludes the proof of the first inequality. The second inequality follows similarly. □

Proof of Proposition 7.2. Since the cokernel of $F_{\mathfrak{I}_0}$ is generated by $\hat{v}\circ\mathfrak{I}_0$, the linear operator defined by $(\hat{\mathfrak{I}},\eta)\mapsto F_{\mathfrak{I}_0}\hat{\mathfrak{I}}+\eta\cdot\hat{v}\circ\mathfrak{I}_0$ is surjective. Hence, there exist a smooth family of right inverses $\mathfrak{S}_t\colon C^{0,\alpha}(P_t,\mathfrak{I}_t^*V\mathfrak{M}_t)\to C^{1,\alpha}(P_t,\mathfrak{I}_t^*V\mathfrak{M}_t)\oplus\mathbf{R}$ to $(\hat{\mathfrak{I}},\eta)\mapsto F_{\mathfrak{I}_t}\hat{\mathfrak{I}}+\eta\hat{v}\circ\mathfrak{I}$. By Proposition 7.15 they are approximate right inverses to $\hat{\mathfrak{K}}_{t,\lambda}$ and can thus be deformed to the desired family of right inverses $\mathfrak{S}_{t,\lambda}$ as in Step 5 of the proof of Proposition 7.1. □

7.5 Estimate of $\mathfrak{p}_{t,\lambda}$ and $\mathfrak{q}_{t,\lambda}$

Proof of Proposition 7.3. The estimates on $\mathfrak{p}_{t,\lambda}$ and $\partial_t\mathfrak{p}_{t,\lambda}$ are immediate consequences of Propositions 5.4 and 7.15 since

$$\mathfrak{p}_{t,\lambda}\hat{\mathfrak{I}}=\rho_{t,\lambda}(L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}}-\iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}}).$$

To estimate $\mathfrak{q}_{t,\lambda}$ we define $\tilde{\pi}_{t,\lambda}:\Omega^1(NP_t,\mathfrak{g}_{E(\mathfrak{I}_{t,\lambda})})\to\Gamma(\mathfrak{I}_t^*V\mathfrak{M}_t)\subset\Omega^1(NP,\mathfrak{g}_{E(\mathfrak{I}_{t,\lambda})})$ by

$$(\tilde{\pi}_{t,\lambda}a)(x):=\sum_\kappa\int_{N_xP}\langle a,\kappa\rangle\,\kappa$$

and set $\tilde{\rho}_{t,\lambda}:=\mathrm{id}-\tilde{\pi}_{t,\lambda}$. One can check that $\tilde{L}_{t,\lambda}$ commutes with $\tilde{\pi}_{t,\lambda}$. Therefore

$$\begin{aligned}\|\mathfrak{q}_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} &\leq\|\pi_{t,\lambda}(L_{t,\lambda}-\tilde{L}_{t,\lambda})\rho_{t,\lambda}a\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}(V_{[0,\sigma),t})}\\ &\quad+\|(\pi_{t,\lambda}-\tilde{\pi}_{t,\lambda})\tilde{L}_{t,\lambda}\rho_{t,\lambda}a\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}(V_{[0,\sigma),t})}\\ &\quad+\|\tilde{\pi}_{t,\lambda}\tilde{L}_{t,\lambda}(\rho_{t,\lambda}-\tilde{\rho}_{t,\lambda})a\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}(V_{[0,\sigma),t})}.\end{aligned}$$

The last two terms are supported in $V_{[\sigma/2,\sigma),t}$ and can easily be seen to be bounded by $c\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}(V_{[0,\sigma),t})}$. The first term can be estimated by

$$c\|\psi_t \wedge [(A_{t,\lambda} - I_{t,\lambda}) \wedge \rho_{t,\lambda}\underline{a}]\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}(V_{[0,\sigma),t})} + \|(\psi_{1,t} + \psi_{\geq 2,t}) \wedge \mathrm{d}_{I_{t,\lambda}}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}(V_{[0,\sigma),t})} + \|(* - *_0)\psi_0 \wedge \mathrm{d}_{I_{t,\lambda}}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}(V_{[0,\sigma),t})}.$$

But this can be bounded by $c\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}}$ using inequalities (6.3), (6.5) and (7.16). This proves the estimate on $\mathfrak{q}_{t,\lambda}$. A similar argument can be used to control $\partial_t\mathfrak{q}_{t,\lambda}$. $\square$

8 Quadratic estimates

Proposition 8.1. *There is a constant $c_Q > 0$ such that for all $\underline{a} \in \mathfrak{A}^{0,\alpha}_{-1,\delta,t,\lambda}$ and all $\hat{\mathfrak{I}} \in C^{0,\alpha}(P_t, \mathfrak{I}_t^* V\mathfrak{M}_t)$ the following holds*

$$\|\rho_{t,\lambda}Q_{t,\lambda}(\underline{a},\hat{\mathfrak{I}})\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \leq c_Q\left(\|\underline{a}\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}} + \|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}\right)^2$$

$$\|\pi_{t,\lambda}Q_{t,\lambda}(\underline{a},\hat{\mathfrak{I}})\|_{C^{0,\alpha}} \leq c_Q\left(\lambda^{-1}\|\underline{a}\|^2_{C^{0,\alpha}_{-1,\delta,t,\lambda}} + \lambda^{-1}\|\underline{a}\|_{C^{0,\alpha}_{-1,\delta,t,\lambda}}\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}} + \|\hat{\mathfrak{I}}\|^2_{C^{0,\alpha}}\right).$$

Moreover, $Q_{t,\lambda}$ depends continuously differentiable on t and continuously on λ and the above estimates also hold with $\partial_t Q_{t,\lambda}$ instead of $Q_{t,\lambda}$.

Proof. The estimates of $\rho_{t,\lambda}Q_{t,\lambda}$ and $\rho_{t,\lambda}\partial_t Q_{t,\lambda}$ are a straight-forward consequence. The only difficulty is in estimating the quadratic contribution of $\hat{\mathfrak{I}}$ to $\pi_{t,\lambda}Q_{t,\lambda}$. But note that

$$\left(*_0[\iota_{t,\lambda}\hat{\mathfrak{I}} \wedge \iota_{t,\lambda}\hat{\mathfrak{I}}] \wedge \psi_{0,t}\right)^{0,1} = 0$$

and hence

$$\|\pi_{t,\lambda} * ([\iota_{t,\lambda}\hat{\mathfrak{I}} \wedge \iota_{t,\lambda}\hat{\mathfrak{I}}] \wedge \psi_t)\|_{C^{0,\alpha}} \leq c\|\hat{\mathfrak{I}}\|^2_{C^{0,\alpha}}$$

using Proposition 5.4 as well as inequalities (6.3) and (6.5). This yields the desired estimate of $\pi_{t,\lambda}Q_{t,\lambda}$. The estimate of $\pi_{t,\lambda}\partial_t Q_{t,\lambda}$ is analogous. $\square$

9 Proof of Theorem 1.1

Proposition 9.1. *For $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ there is a constant $c > 0$ as well as $\underline{a}(t,\lambda) \in \mathfrak{A}^{1,\alpha}_{-1,\delta,t,\lambda}$, $\hat{\mathfrak{I}}(t,\lambda) \in C^{1,\alpha}(P_t, \mathfrak{I}_t^* V\mathfrak{M}_t)$ and $\eta(t,\lambda) \in \mathbf{R}$ depending continuously differentiable on t and continuously on λ such that the connection $\tilde{A}_{t,\lambda} := A_{t,\lambda} + a(t,\lambda) + \iota_{t,\lambda}\hat{\mathfrak{I}}$ satisfies*

$$* \left(F_{\tilde{A}_{t,\lambda}} \wedge \psi_t\right) + \mathrm{d}_{\tilde{A}_{t,\lambda}}\xi(t,\lambda) + (\mu(t) + \eta(t,\lambda)) \cdot \iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t = 0. \tag{9.2}$$

and

$$\|\underline{a}(t,\lambda)\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}} \leq c\lambda^{2+\delta/2} \qquad \|\hat{\mathfrak{I}}(t,\lambda)\|_{C^{1,\alpha}} \leq c\lambda \qquad |\eta(t,\lambda)| + |\partial_t\eta(t,\lambda)| \leq c\lambda.$$

Proof. In the notation of Sections 4 and 6 we can write equation (9.2) as

$$\mathfrak{L}_{t,\lambda}\underline{a} + \mathfrak{p}_{t,\lambda}\hat{\mathfrak{I}} + \rho_{t,\lambda}Q_{t,\lambda}(\underline{a},\hat{\mathfrak{I}}) + \rho_{t,\lambda}\mathfrak{e}_{t,\lambda} = 0$$
$$\mathfrak{K}_{t,\lambda}\hat{\mathfrak{I}} + \mathfrak{q}_{t,\lambda}\underline{a} + \pi_{t,\lambda}Q_{t,\lambda}(\underline{a},\hat{\mathfrak{I}}) + \eta\cdot\hat{v}\circ\mathfrak{I}_t + \pi_{t,\lambda}\mathfrak{e}_{t,\lambda} = 0.$$

If we make the ansatz

$$\underline{a} = \mathfrak{R}_{t,\lambda}\underline{b} \qquad (\hat{\mathfrak{I}},\eta) = \mathfrak{S}_{t,\lambda}\hat{\mathfrak{I}},$$

then this becomes

$$\underline{b} + \mathfrak{p}_{t,\lambda}\hat{\mathfrak{I}} + \rho_{t,\lambda}Q(\underline{a},\hat{\mathfrak{I}}) + \rho_{t,\lambda}\mathfrak{e}_{t,\lambda} = 0$$
$$\hat{\mathfrak{I}} + \mathfrak{q}_{t,\lambda}\underline{a} + \pi_{t,\lambda}Q(\underline{a},\hat{\mathfrak{I}}) + \pi_{t,\lambda}\mathfrak{e}_{t,\lambda} = 0.$$

We can write this as a fixed point equation

$$(\underline{b},\hat{\mathfrak{I}}) = \mathfrak{T}_{t,\lambda}(\underline{b},\hat{\mathfrak{I}})$$

with

$$\mathfrak{T}_{t,\lambda}(\underline{b},\hat{\mathfrak{I}}) := -\begin{pmatrix}\mathfrak{p}_{t,\lambda}\hat{\mathfrak{I}} + \rho_{t,\lambda}Q(\underline{a},\hat{\mathfrak{I}}) + \rho_{t,\lambda}\mathfrak{e}_{t,\lambda}\\ \mathfrak{q}_{t,\lambda}\underline{a} + \pi_{t,\lambda}Q(\underline{a},\hat{\mathfrak{I}}) + \pi_{t,\lambda}\mathfrak{e}_{t,\lambda}\end{pmatrix}.$$

Now, consider

$$\mathfrak{X}_{t,\lambda} = \left\{(\underline{b},\hat{\mathfrak{I}}) : \|\underline{b}\| \le 2c_{\mathfrak{e}}\lambda^{2+\delta/2}, \|\hat{\mathfrak{I}}\| \le 2c_{\mathfrak{e}}\lambda\right\}$$

as a subspace of $\mathfrak{A}^{0,\alpha}_{-2,\delta,t,\lambda} \oplus C^{0,\alpha}(P_t,\mathfrak{I}_t^*V\mathfrak{M}_t)$ equipped with the norm

$$\|(\underline{b},\hat{\mathfrak{I}})\| := \max\{\|\underline{b}\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}}, \lambda\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}\}.$$

Using Propositions 6.1, 7.1, 7.2, 7.3 and 8.1 it is easy to see that $\mathfrak{T}_{t,\lambda}$ defines a contraction from $\mathfrak{X}_{t,\lambda}$ into itself. We define $(\underline{b},\hat{\mathfrak{I}})$ to be the unique fixed point of $\mathfrak{T}_{t,\lambda}$ in $\mathfrak{X}_{t,\lambda}$. Hence, in particular,

$$\|\underline{b}(t,\lambda)\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le c\lambda^{2+\delta/2} \qquad \|\hat{\mathfrak{I}}(t,\lambda)\|_{C^{0,\alpha}} \le c\lambda$$

Since $\mathfrak{T}_{t,\lambda}$ depends continuously differentiable on t and continuously on λ so does its fixed point. Differentiating the fixed point equation leads to

$$\partial_t(\underline{b},\hat{\mathfrak{I}}) = (\partial_t\mathfrak{T}_{t,\lambda})(\underline{b},\hat{\mathfrak{I}}) + \mathfrak{T}_{t,\lambda}\partial_t(\underline{b},\hat{\mathfrak{I}}).$$

The term $(\partial_t\mathfrak{T}_{t,\lambda})(\underline{b},\hat{\mathfrak{I}})$ can be controlled by Propositions 7.1, 7.2, 7.3 and 8.1 and a slight modification of the above argument shows that

$$\|\partial_t\underline{b}(t,\lambda)\|_{C^{0,\alpha}_{-2,\delta,t,\lambda}} \le c\lambda^{2+\delta/2} \qquad \|\partial_t\hat{\mathfrak{I}}(t,\lambda)\|_{C^{0,\alpha}} \le c\lambda.$$

Therefore

$$\|\underline{a}(t,\lambda)\|_{C^{1,\alpha}_{-1,\delta,t,\lambda}} \le c\lambda^{2+\delta/2} \qquad \|\hat{\mathfrak{I}}(t,\lambda)\|_{C^{1,\alpha}} \le c\lambda \qquad |\eta(t,\lambda)| + |\partial_t\eta(t,\lambda)| \le c\lambda$$

as desired. □

Proof of Theorem 1.1. Because of Proposition 9.1 the problem of finding A_λ is reduced to constructing $t\colon [0,\Lambda] \to (-T',T')$ such that $t(0) = 0$ and

$$\mu(t(\lambda)) + \eta(t(\lambda),\lambda) = 0$$

for $t \in (0,\Lambda]$. But this is an immediate consequence of the implicit function theorem, because $\mu(0) = 0$, $\partial_t\mu(0) \neq 0$ and $|\eta| + |\partial_t\eta| \le c\lambda$. The resulting connection $A_\lambda = \tilde{A}_{t(\lambda),\lambda}$ will be smooth by elliptic regularity. That A_λ converges to A_0 on the complement of P_0 and that at each point $x \in P_0$ an ASD instanton in the equivalence class of $\mathfrak{I}(x)$ bubbles of transversely is clear, since we constructed $A_{t,\lambda}$ accordingly and A_λ is a small perturbation of $A_{t(\lambda),\lambda}$. □

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